Power series

In the previous section we have seen that the function $\exp(x)$ which is defined by the series $\sum_{k=0}^{\infty} x^k/k!$ is a continuous function for $|x| < 1$. Much earlier we examined the Geometric series $\sum_{k=0}^{\infty} x^k$ which is a rather complicated way of writing $1/(1-x)$ for $0 < x < 1$. We can ask more generally when such series define continuous functions.

A series of the form $\sum_{k=0}^{\infty} a_k x^k$ is called a (formal) "power series". The partial sums $f_n(x) = \sum_{k=0}^{n} a_k x^k$ are polynomials. Hence, the $f_n(x)$ are continuous functions. So, if we can show that the sequence $(f_n(x))_{n\geq 1}$ is uniformly convergent for x lying in some interval $[a, b]$, then, as before, we will obtain a continuous function in that interval represented by this power series.

Geometric Series

The base example for convergence of power series is the geometric series $\sum_{k=0}^{\infty} x^k$. The partial sums of this series are based on the identity (which we have seen earlier)

$$f_m(x) = \sum_{k=0}^{m} x^k = \frac{1 - x^{m+1}}{1-x} = \frac{1}{1-x} - \frac{x^{m+1}}{1-x}$$

which makes sense for $x \neq 1$. When is this uniformly convergent to $f(x) = 1/(1-x)$?

If we have $|x| \leq r < 1$, we see that the "error term" is

$$\left| \frac{x^{m+1}}{1-x} \right| \leq \frac{r^{m+1}}{1-r}$$

We have seen that $(r^n)_{n\geq 1}$ converges to 0. So, given a positive integer p, we can choose m_p so that, for $m \geq m_p$ we have $r^{m+1}/(1-r) < 1/p$. It follows that $|f(x) - f_m(x)| < 1/p$ for $m \geq m_p$ *for all* x such that $|x| \leq r$.

Hence, we have proved uniform convergence of $(f_m)_{m\geq 1}$ to f for $|x| \leq r < 1$. Clearly this proof works for *all* r such that $0 < r < 1$.

Bounded sequences

Given a sequence $(a_n)_{n\geq 1}$ in which all the terms are bounded, let us examine the series $\sum_{n=1} a_n x^n$. Since there is an M so that, for all n we have

$$|a_n| < M$$

As before we take $f_m(x) = \sum_{k=0}^{m} a_k x^k$ and ask whether the sequence $(f_m)_{m\geq 1}$ is uniformly convergent.

The difference with the previous case is that we no longer can "guess" the limiting function. So, we must use Cauchy's version of uniform convergence. So we calculate for $n \geq m$ that

$$|f_n(x) - f_m(x)| = \left| \sum_{k=m+1}^{n} a_k x^k \right|$$
$$\leq \sum_{k=m+1}^{n} \left| a_k x^k \right| \leq \sum_{k=m+1}^{n} M|x|^k$$
$$\leq \sum_{k=m+1}^{\infty} M|x|^k = \frac{M|x|^{m+1}}{1-|x|}$$

Note that the last two steps make sense *only* for $|x| < 1$. Now, as above, let us assume that $|x| \leq r < 1$. Then, we get

$$|f_n(x) - f_m(x)| \leq \frac{Mr^{m+1}}{1-r}$$

Since $r < 1$, the sequence $(r^n)_{n \geq 1}$ converges to 0. So, given a positive integer p, there is an m_p, so that for all $m \geq m_p$, we have $(Mr^{m+1})/(1-r) < 1/p$. We then get $|f_n(x) - f_m(x)| < 1/p$ for all $n \geq m \geq m_p$ and *for all* x such that $|x| \leq r < 1$.

Hence we have demonstrated that Cauchy's criterion for uniform convergence is satisfied by the sequence $(f_n)_{n \geq 1}$ in the region $|x| \leq r < 1$. This argument works for every r such that $0 < r < 1$, so the limit function f of this sequence of polynomials is a continuous function in $|x| < 1$.

Convergent sequences

Since a convergent sequence is automatically bounded, a special case of this is when $(a_n)_{n \geq 1}$ is a sequence that *converges*. In particular, we see that $\sum_{n=1}^{\infty} (-1)^{n+1}(x^n/n)$ converges uniformly for $|x| \leq r < 1$ and defines a continuous function for $|x| < 1$. We will examine the properties of the function defined by this series later on.

Reverse implication

Given a power series $\sum_{n=0}^{\infty} a_n x^n$. *Suppose* that there is some non-zero number t so that $\sum_{n=0}^{\infty} a_n t^n$ is a convergent series.

Since $\sum_{n=0}^{\infty} a_n t^n$ is convergent the *sequence* $(a_n t^n)_{n \geq 1}$ converges to 0.

It follows from the earlier discussion that the power series $\sum_{n=0}^{\infty} a_n t^n y^n$ converges uniformly for $|y| \leq r < 1$ and defines a continuous function for $|y| < 1$. Using the variable $x = ty$, this condition becomes $|x| \leq r|t| < |t|$. Since $0 < r < 1$ is arbitrary, the result number $s = r|t|$ is an arbitrary number such that $0 < s < |t|$.

In other words, the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly for $|x| \leq s < |t|$ and gives a continuous function for $|x| < |t|$.

Applying this to the series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ which we have seen as a convergent series, we get again the convergence of $\sum_{n=1}^{\infty}(-1)^{n+1}x^n/n$ for $|x| < 1$. Note that this series is *not* convergent for $x = -1$, so this limitation on $|x|$ is precise!

Convergence

The above reasoning suggests that in order to obtain convergence of the series we need to compare the series with the Geometric series. Suppose that for some positive number R, there is an integer n_R and a positive number M so that

$$|a_k| \leq \frac{M}{R^k}$$

for all $k \geq n_R$. In that case for all x satisfying $|x| \leq r < R$ and for any $n \geq n_R$, the geometric series gives an upper bound

$$\sum_{k=n}^{\infty}|a_k x^k| \leq \sum_{k=n}^{\infty} M \left(\frac{|x|}{R}\right)^k = M \left(\frac{|x|}{R}\right)^n \frac{1}{1-\frac{|x|}{R}} \leq M \left(\frac{r}{R}\right)^n \frac{1}{1-\frac{r}{R}}$$

Since $\frac{r}{R} < 1$, given *any* positive integer p, we can choose n_p large enough so that

$$M \left(\frac{r}{R}\right)^{n_p} \frac{1}{1-\frac{r}{R}} < 1/p$$

Now, consider the difference between the partial sums for $m \geq n \geq n_p$,

$$|f_m(x) - f_n(x)| \leq \sum_{k=n+1}^{m} |a_k x^k| \leq \sum_{k=n_p}^{\infty} |a_k x^k| < 1/p$$

Since this can be done for any p, we see that $(f_n)_{n\geq 1}$ is a sequence that satisfies Cauchy's criterion for uniform convergence. As seen above this means that the sequence $(f_n)_{n\geq 1}$ converges uniformly on $|x| \leq r < R$. Since the f_n's are polynomials and $0 < r < R$ is arbitrary, the limit function f is continuous in the region $|x| < R$.

In summary, we have shown that if, for some positive $R > 0$, there is a positive constant M and an integer n_R such that, for all $k \geq n_R$ we have

$$|a_k| < \frac{M}{R^k}$$

then the series $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly in $|x| \leq r < R$ for arbitrary r such that $0 < r < R$ and defines a continuous function in the region $|x| < R$.

We note that the condition that for all $k \geq n_R$ we have

$$|a_k| < \frac{M}{R^k}$$

can be seen as the condition that $(|a_n|)_{n\geq 1}$ is *dominated* by the sequence $(M/R^n)_{n\geq 1}$. This underlines the importance of the comparison of the growth of sequences that we studied during the first few lectures.

The exponential series

Once again let us examine the series $\sum_{n=0}^{\infty} x^k/k!$. Given *any* positive R, we have

$$\frac{1}{n!} = \frac{R^n}{n!}\frac{1}{R^n} = \frac{R^p}{p!} \cdot \frac{R^{n-p}}{(p+1)\cdot(p+2)\cdots n}\frac{1}{R^n}$$

Now, if n_R is an (or the smallest) integer greater than R, then for $p = n_R$ we have $R < p + q$ for positive integers q and so

$$\frac{R^{n-p}}{(p+1)\cdot(p+2)\cdots n} < 1$$

We then put $M = (R^p)/p!$ and obtain, for all $n \geq n_R$

$$\frac{1}{n!} = \frac{R^p}{p!} \cdot \frac{R^{n-p}}{(p+1)\cdot(p+2)\cdots n}\frac{1}{R^n} < M\frac{1}{R^n}$$

as required.

In other words, the series $\sum_{n=0}^{\infty} x^k/k!$ converges to a continuous function $\exp(x)$ in $|x| < R$ for *all* choices of positive R. This makes it define a continuous function for all values of x.

Sine and Cosine series

The Sine, Cosine, Hyperbolic Sine and Hyperbolic Cosine series are given by

$$\begin{aligned}
\sin(x) &= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\\
\cos(x) &= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\\
\sinh(x) &= \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}\\
\cosh(x) &= \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}
\end{aligned}$$

We note that these series are obtained by dropping alternate terms of the series $\exp(x)$ and possibly putting a sign. Looked at that way, it is obvious that the same argument as given above will show that these series also converge and give continuous functions for $|x| < R$ for *all* positive values of R. Thus, these are continuous functions for all values of x.