Solutions to End-Sem Exam

1. Give the number or the example as indicated below.

(1 mark) (a) The number of units in the ring $\mathbb{Z}/36\mathbb{Z}$.

> **Solution:** By the Chinese Remainder Theorem, we need to calculate the number of units in $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/9\mathbb{Z}$ and multiply these. In the first case, there are $2 = 4 - 2$ units and in the second case there are $6 = 9 - 3$ units. So all in all there are $12 = 2 \times 6$ units.

(1 mark) (b) The number of elements k in the ring $\mathbb{Z}/36\mathbb{Z}$ for which $6k = 0$.

> **Solution:** We need to solve $6k = 36m$ or equivalently $k = 6m$ in this ring. This consists of $\{0, 6, 12, 18, 24, 30\}$ or 6 elements.

(1 mark) (c) An example of an idempotent in the ring $\mathbb{Z}/30\mathbb{Z}$ which is different from 0 and 1.

> **Solution:** By the Chinese Remainder Theorem, we need to look at elements that are 0 or 1 in $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$. This approach gives us the elements $\{0, 1, 6, 15, 10, 21, 25, 16\}$. So any of the last six is a permissible answer. Note that $6^2 = 36 = 6 \pmod{30}$ so that is an "easy" answer.

(1 mark) (d) An example of a non-zero prime ideal in the ring $\mathbb{Q}[T]$.

> **Solution:** We need an irreducible polynomial over $\mathbb{Q}$. The polynomial T is such an example.

(1 mark) (e) An example of an idempotent element in the ring $\mathbb{Q}[T]/(T^3 - T)$ which is different from 0 and 1.

> **Solution:** By the Chinese Remainder Theorem, we need to look at elements that are 0 or 1 in $\mathbb{Q}[T]/T$, $\mathbb{Q}[T]/(T-1)$ and $\mathbb{Q}[T]/(T+1)$. These elements are $\{0, 1, (T^2 - T)/2, (T^2 + T)/2, 1 - T^2, T^2, 1 + (-T^2 - T)/2, 1 + (-T^2 + T)/2\}$. So any of the last six is a permissible answer. Note that $T^4 = T(T^3) = T^2$, so that is an "easy" answer.

(1 mark) (f) An example of a quaternion which does not commute with $\hat{\imath}$.

> **Solution:** Any quaternion that has a non-zero component of the form $a\hat{\jmath} + b\hat{k}$ is such a quaternion. For example $\hat{k}$.

(1 mark) (g) An example of an orthogonal 2×2 matrix with determinant -1.

Solution: This is a reflection matrix. For example $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(1 mark) (h) An example of a 2×2 unitary matrix for which at least one entry is not real.

Solution: Any diagonal matrix with diagonal entries of absolute value 1 is unitary. Hence, $\begin{pmatrix} \iota & 0 \\ 0 & 1 \end{pmatrix}$ is such a matrix.

(1 mark) (i) An example of a matrix over $\mathbb{C}$ which cannot be diagonalised.

Solution: Any non-zero nilpotent matrix cannot be diagonalised. Hence, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is such a matrix.

(1 mark) (j) An example of a quadratic form which is *not* positive-semi-definite.

Solution: Any diagonal form which has a negative term is such a quadratic form. Hence $q(x) = -x^2$ is such a form.

(5 marks) 2. Write down (upto isomorphism) all the possible abelian groups of order 72 that are generated by at most 3 elements. (Hint: What are all the Smith normal matrices A so that the group is $\mathbb{Z}^3/A\mathbb{Z}^3$?)

Solution: As the hint says, we need to write all diagonal 3×3 matrices with diagonal entries a, b, c so that a divides b and b divides c and $abc = 72$. We write $b = da$ and $c = eb$. So we have $a^3d^2e = 72$. Since a^3 divides 72, we must have $a = 1$ or $a = 2$. since d^2 divides 72, we must have $d = 1, 2, 3, 6$. So we have the solutions

$$(a, d, e) \in \{(1, 1, 72), (1, 2, 18), (1, 3, 8), (1, 6, 2), (2, 1, 9), (2, 3, 1)\}$$

or equivalently,

$$(a, b, c) \in \{(1, 1, 72), (1, 2, 36), (1, 3, 24), (1, 6, 12), (2, 2, 18), (2, 6, 6)\}$$

Hence there are 6 such groups.

$$\mathbb{Z}/72\mathbb{Z}$$
$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}$$
$$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/24\mathbb{Z}$$
$$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$$
$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z}$$
$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$$

(10 marks) 3. Using row and column reduction, find the matrices S and T with *integer* coefficients so that SAT is a diagonal matrix in Smith normal form. (Hint: Be systematic and apply the pivoting method.)

$$A = \begin{pmatrix} 6 & 2 & 3 & 0 \\ 8 & 2 & 2 & 2 \\ 8 & 2 & 2 & 5 \\ 4 & 0 & 4 & -12 \end{pmatrix}$$

Solution: We note that column operations correspond to multiplication by elementary matrices on the right and row operations correspond to multiplication by elementary matrices on the left. We first subtract 3 times column 2 from column 1 and also subtract column 2 from column 3

$$AT_1 = \begin{pmatrix} 6 & 2 & 3 & 0 \\ 8 & 2 & 2 & 2 \\ 8 & 2 & 2 & 5 \\ 4 & 0 & 4 & -12 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 0 & 5 \\ 4 & 0 & 4 & -12 \end{pmatrix}$$

Next we subtract 2 times column 3 from column 2

$$AT_1T_2 =$$

$$A \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 0 & 5 \\ 4 & 0 & 4 & -12 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 0 & 5 \\ 4 & -8 & 4 & -12 \end{pmatrix}$$

We now subtract row 1 from row 4

$S_1A(T_1T_2) =$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{pmatrix} A \begin{pmatrix}1 & 0 & 0 & 0\\ -3 & 3 & -1 & 0\\ 0 & -2 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{pmatrix} \begin{pmatrix}0 & 0 & 1 & 0\\ 2 & 2 & 0 & 2\\ 2 & 2 & 0 & 5\\ 4 & -8 & 4 & -12\end{pmatrix} =$$

$$\begin{pmatrix}0 & 0 & 1 & 0\\ 2 & 2 & 0 & 2\\ 2 & 2 & 0 & 5\\ 4 & -8 & 0 & -12\end{pmatrix}$$

Now we subtract column 2 from column 1 and also column 2 from column 4

$S_1A(T_1T_2)T_3 =$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{pmatrix} A \begin{pmatrix}1 & 0 & 0 & 0\\ -3 & 3 & -1 & 0\\ 0 & -2 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}0 & 0 & 1 & 0\\ 2 & 2 & 0 & 2\\ 2 & 2 & 0 & 5\\ 4 & -8 & 0 & -12\end{pmatrix} \begin{pmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 2 & 0 & 0\\ 0 & 2 & 0 & 3\\ 12 & -8 & 0 & -4\end{pmatrix}$$

Now we subtract row 2 from row 3 and add 4 times row 2 to row 4

$$S_2S_1A(T_1T_2T_3) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&4&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-1&0&0&1\end{pmatrix}A\begin{pmatrix}1&0&0&0\\-6&3&-1&-3\\2&-2&1&2\\0&0&0&1\end{pmatrix} =$$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&4&0&1\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&2&0&0\\0&2&0&3\\12&-8&0&-4\end{pmatrix} =$$

$$\begin{pmatrix}0&0&1&0\\0&2&0&0\\0&0&0&3\\12&0&0&-4\end{pmatrix}$$

Now we add row 3 to row 4

$$S_3(S_2S_1)A(T_1T_2T_3) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\-1&4&0&1\end{pmatrix}A\begin{pmatrix}1&0&0&0\\-6&3&-1&-3\\2&-2&1&2\\0&0&0&1\end{pmatrix} =$$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&2&0&0\\0&0&0&3\\12&0&0&-4\end{pmatrix} =$$

$$\begin{pmatrix}0&0&1&0\\0&2&0&0\\0&0&0&3\\12&0&0&-1\end{pmatrix}$$

Now we add 3 times row 4 to row 1

$$S_4(S_3S_2S_1)A(T_1T_2T_3) =$$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ -1 & 3 & 1 & 1\end{pmatrix} A \begin{pmatrix}1 & 0 & 0 & 0\\ -6 & 3 & -1 & -3\\ 2 & -2 & 1 & 2\\ 0 & 0 & 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 3\\ 12 & 0 & 0 & -1\end{pmatrix} =$$

$$\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 2 & 0 & 0\\ 36 & 0 & 0 & 0\\ 12 & 0 & 0 & -1\end{pmatrix}$$

Now we add 12 times column 4 to column 1

$$(S_4S_3S_2S_1)A(T_1T_2T_3)T_4 =$$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -3 & 9 & 4 & 3\\ -1 & 3 & 1 & 1\end{pmatrix} A \begin{pmatrix}1 & 0 & 0 & 0\\ -6 & 3 & -1 & -3\\ 2 & -2 & 1 & 2\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 12 & 0 & 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 2 & 0 & 0\\ 36 & 0 & 0 & 0\\ 12 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 12 & 0 & 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 2 & 0 & 0\\ 36 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

Finally, we only need to permute the rows and columns to get the diagonal form.

We use P and Q to denote the appropriate permutation.

$$P(S_4S_3S_2S_1)A(T_1T_2T_3T_4)Q =$$
$$\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\-3&9&4&3\\-1&3&1&1\end{pmatrix}A\begin{pmatrix}1&0&0&0\\-42&3&-1&-3\\26&-2&1&2\\12&0&0&1\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix} =$$
$$\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&2&0\\0&0&0&36\end{pmatrix}$$

Finally, this gives us

$$(PS_4S_3S_2S_1)A(T_1T_2T_3T_4Q) =$$
$$\begin{pmatrix}1&0&0&0\\-1&3&1&1\\0&1&0&0\\-3&9&4&3\end{pmatrix}A\begin{pmatrix}0&0&0&1\\-3&-1&3&-42\\2&1&-2&26\\1&0&0&12\end{pmatrix} =$$
$$\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&2&0\\0&0&0&36\end{pmatrix}$$

4. Consider the quadratic form $q(x, y) = x^2 - 2xy$.

(1 mark) (a) Convert the quadratic form $q(x, y)$ to diagonal form.

Solution: We have
$$x^2 - 2xy = (x - y)^2 - y^2$$

(1 mark) (b) Find the matrix A so that $q(x, y) = v^t Av$ where $v = \begin{pmatrix}x\\y\end{pmatrix}$.

Solution: We write this is $q(x, y) = xx - xy - yx + 0yy$ so that the matrix is easily seen to be
$$A = \begin{pmatrix}1&-1\\-1&0\end{pmatrix}$$

(1 mark) (c) Find the vector (x, y) on the unit circle where $q(x, y)$ takes its maximum value.

(1 mark) (d) Find the vector (x, y) on the unit circle where $q(x, y)$ takes its minimum value.

Solution: (For both parts above). The minimum and maximum values correspond to eigen-vectors of A. We calculate the characteristic polynomial of A as $(1-T)(-T)-(-1)(-1)$ or equivalently T^2-T-1. The roots of this polynomial are $T = (1 \pm \sqrt{5})/2$. Put $\phi = (1+\sqrt{5})/2$ as one eigenvalue, then the other eigenvale is $1-\phi$. Moreover, $\phi^2 = 1+\phi$. We thus see that

$$\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -\phi \\ 1 \end{pmatrix} = \begin{pmatrix} -1-\phi \\ \phi \end{pmatrix} = \phi \begin{pmatrix} -\phi \\ 1 \end{pmatrix}$$

Thus, $(-\phi, 1)$ is an eigenvector with eigenvalue ϕ.

Since the matrix is symmetric, the vector orthogonal to this is also an eigenvector. In other words $(1, \phi)$ is also an eigenvector with eigenvalue $1-\phi$.

To be on the unit circle, the vectors have to have unit length. Note that $1+\phi^2 = 2+\phi$. We have the maximum value at the vector with eigenvalue ϕ

$$\frac{1}{\sqrt{2+\phi}} \begin{pmatrix} -\phi \\ 1 \end{pmatrix}$$

and the minimum value at the vector with eigenvalue $1-\phi$

$$\frac{1}{\sqrt{2+\phi}} \begin{pmatrix} 1 \\ \phi \end{pmatrix}$$

(1 mark) (e) Find an orthogonal matrix S so that SAS^{-1} is a diagonal matrix.

Solution: The matrix S^{-1} is the matrix of the form $[v_1, v_2]$ where v_i are the unit eigenvectors. In other words,

$$S^{-1} = \frac{1}{\sqrt{2+\phi}} \begin{pmatrix} -\phi & 1 \\ 1 & \phi \end{pmatrix}$$

It follows that $\det(S^{-1}) = -1$ and so (why?)

$$S = S^{-1} = \frac{1}{\sqrt{2+\phi}} \begin{pmatrix} -\phi & 1 \\ 1 & \phi \end{pmatrix}$$

Note that S is not unique, for example, we can also interchange the columns of S or take $-S$.

(5 marks) 5. Decompose the following matrix into *one* of the forms KAK, KAN or KP.

$$G = \begin{pmatrix} \iota & 1 \\ 1 & \iota \end{pmatrix}$$

Solution: We calculate

$$G^*G = \begin{pmatrix} -\iota & 1 \\ 1 & -\iota \end{pmatrix} \begin{pmatrix} \iota & 1 \\ 1 & \iota \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

The positive square root of this matrix is

$$P = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

It follows that

$$K = G \cdot P^{-1} = \frac{1}{\sqrt{2}} G$$

is a unitary matrix. Moreover, we see that P is a scalar matrix hence it commutes with anything and is also a diagonal matrix with positive diagonal entries; so we can also put $A = P$. We put, $N = 1$ and $U = K$ and $V = 1$

$$G = K \cdot P = K \cdot A \cdot N = U \cdot A \cdot V$$

All the decompositions are the same!

6. Given an $n \times n$ matrix A over the field $\mathbb{C}$ of complex numbers. Recall that an eigen-vector is a non-zero column-n vector v so that $A \cdot v = a \cdot v$ for some complex number a. Recall that a number a for which there *exists* such a non-zero vector is called an eigen-vector is called an eigen-value.

(1 mark) (a) If $P(T)$ is a polynomial with coefficients in $\mathbb{C}$, then show that $P(A) \cdot v = P(a)v$. (Hint: First check for $P(T) = T^n$.)

Solution: We note, by induction on k that $A^k \cdot v = a^k v$. For, this is given for $k = 1$ and if we have shown it for k, then we have

$$A^{k+1} \cdot v = A \cdot (A^k \cdot v) = A \cdot (a^k v) = a^k (A \cdot v) = a^k (av) = a^{k+1} v$$

Since $P(A)$ is a linear combination of terms of the form $c_k A^k$, the result follows.

(1 mark) (b) If $Q_A(T)$ is the minimal polynomial of A, then show that $Q(a) = 0$ for any eigen-value a of A.

Solution: Since $Q_A(T)$ is the minimal polynomial of A, we have $Q_A(A) = 0$. From the previous exercise, it follows that $Q_A(A) \cdot v = Q_A(a)v$. Hence $Q_A(a)v = 0$. Since $v \neq 0$, it follows that $Q_A(a) = 0$.

(1 mark) (c) Show that a is an eigen-value of A if and only if $\det(A - a \cdot 1) = 0$. (Hint: Recall that $B \cdot w = 0$ for a non-zero vector w and an $n \times n$ matrix B if and only if $\det(B) = 0$.)

> **Solution:** We note that $A \cdot v = av$ if and only if $(A - a \cdot 1) \cdot v =$. Now, $(A - a \cdot 1) \cdot v = 0$ has a solution with $v \neq 0$ if and only $\det(A - a \cdot 1) = 0$. This shows that a is an eigen-value of A if and only if $\det(A - a \cdot 1) = 0$.

(2 marks) (d) Show that the polynomial $\det(A - T \cdot 1)$ divides $Q_A(T)^N$ for every large enough integer N. (Hint: Use fundamental theorem of algebra.)

> **Solution:** From what has been shown above, *every* solution a of $\det(A - a \cdot 1) = 0$ also satisfies $Q_A(a) = 0$. Since $\mathbb{C}$ is algebraically closed, $P(T) = \det(A - T \cdot 1)$ is a product of terms of the form $(a_i - T)^{n_i}$ (upto a non-zero constant factor) where a_i are the roots of the polynomial $P(T)$. Each of factors $(a_i - T)$ divides $Q_A(T)$ since any root of $P(T)$ is a root of $Q_A(T)$. Hence, if we take N to be greater than the maximum of the numbers n_i, then $P(T)$ divides $Q_A(T)^N$.

7. Recall that a matrix A over a field F is said to be diagonalisable *over that field* if there is a basis of F^n consisting of eigen-vectors for A. In what follows, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{F}_2$ denotes the field with two elements and $\mathbb{C}$ denotes the field of complex numbers.

   Indicate which of the following statements are true. If the statement is not true provide an example showing that it is false.

(1 mark) (a) If a matrix A over $\mathbb{Q}$ satisfies $A^2 = 1$, then it is diagonalisable over $\mathbb{Q}$.

> **Solution:** Since the roots of the polynomial $T^2 - 1$ are 1 and -1 and are distinct, so the matrix is diagonalisable over $\mathbb{Q}$.

(1 mark) (b) If a matrix A over $\mathbb{F}_2$ satisfies $A^2 = 1$, then it is diagonalisable over $\mathbb{F}_2$.

> **Solution:** Since $1 = -1$ in $\mathbb{F}_2$, the previous argument does not work! We can take the matrix of multiplication by T on $\mathbb{F}_{\not{2}}[T]/(T^2 - 1)$ in the basis $\{1, T - 1\}$. This is $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. We check easily that $A^2 = A$ over $\mathbb{F}_2$.

(1 mark) (c) If a matrix A over $\mathbb{Q}$ satisfies $A^3 = 1$, then it is diagonalisable over $\mathbb{Q}$.

> **Solution:** Since the polynomial $T^3 - 1$ does not have *all* its roots in $\mathbb{Q}$, the

matrix *need not be* diagonalisable over $\mathbb{Q}$. For example, we can take

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The vector $v = (1, 1, 1)$ is fixed. In the plane $v^{\perp}$ perpendicular to v, this is a rotation by $2\pi/3$ and so it has no other eigenvectors with entries in $\mathbb{Q}$ (or even $\mathbb{R}$!).

(1 mark) (d) If a matrix A over $\mathbb{Q}$ satisfies $A^3 = 1$, then it is *not* diagonalisable over $\mathbb{Q}$.

Solution: The identity matrix $A = 1$ satisfies $A^3 = 1$ and it *is* diagonal! So there is a matrix A for which the above statement is false.

(1 mark) (e) If a matrix A over $\mathbb{C}$ satisfies $A^n = 1$, for some integer n, then it is diagonalisable over $\mathbb{C}$.

Solution: The polynomial $T^n - 1 = 0$ has distinct roots in $\mathbb{C}$. Since the minimal polynomial of A divides this polynomial, we see that the minimal polynomial of A has distinct roots. Hence, A is diagonalisable over $\mathbb{C}$.

8. Consider the $\mathbb{Q}[T]$ module $V = \mathbb{Q}[T]/(T^2 - T) \times \mathbb{Q}[T]/(T^2)$.

(1 mark) (a) Calculate the matrix A of multiplication by T on the vector space V in a suitable basis.

Solution: We use the basis

$$\{(1, 0), (T, 0), (0, 1), (0, T)\}$$

This gives us the matrix (by simple calculation)

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(1 mark) (b) Calculate the characteristic polynomial this matrix A.

Solution: The characteristic polynomial $\det(A - T \cdot 1)$ is

$$\det \begin{pmatrix} -T & 0 & 0 & 0 \\ 1 & 1-T & 0 & 0 \\ 0 & 0 & -T & 0 \\ 0 & 0 & 1 & -T \end{pmatrix}$$

We see that this is

$$(-T)(1-T)(-T)^2 = T^3(T-1) = T^4 - T^3$$

(1 mark) (c) Calculate the minimal polynomial this matrix A.

Solution: Since multiplication by this polynomial should be 0 on each factor of V, it is the least common multiple of $T^2 - T$ and T^2. In other words it is $T^2(T-1) = T^3 - T^2$.

(1 mark) (d) What is the Smith normal form of the matrix $A - T$ for a variable T.

Solution: The Smith normal form has diagonal entries $P_1(T)$, $P_2(T)$, $P_3(T)$, $P_4(T)$ where $P_4(T)$ is the minimal polynomial and the product of these entries is the characteristic polynomial. Since

$$T^4 - T^3 = T \cdot (T^3 - T^2)$$

we see that we must have $P_1(T) = P_2(T) = 1$ and $P_3(T) = T$. Thus, the Smith normal form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & T & 0 \\ 0 & 0 & 0 & (T^3 - T^2) \end{pmatrix}$$

(1 mark) (e) What is the Jordan form of A?

Solution: The Jordan form of A is a block form made by look at the matrix of multiplication by T on $\mathbb{Q}[T]/(P_i(T)$ for each i and further factoring the $P_i(T)$. The only polynomial that needs to be factored is $T^3 - T^2 = T^2(T-1)$. It follows that the Jordan form of A is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$