

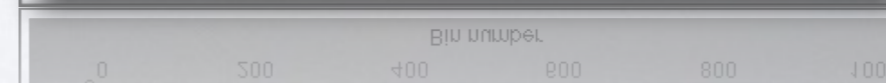
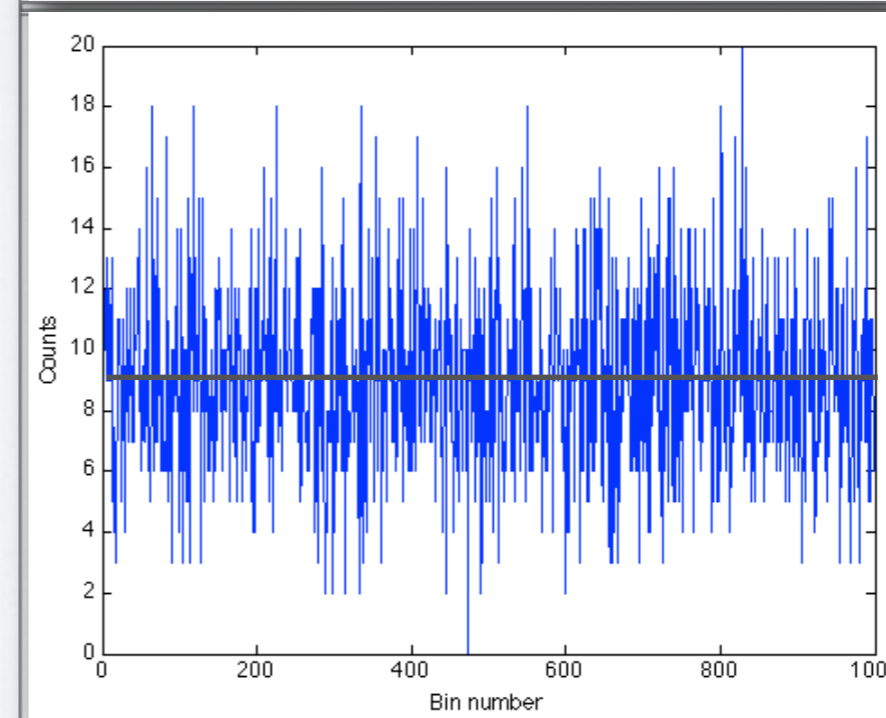
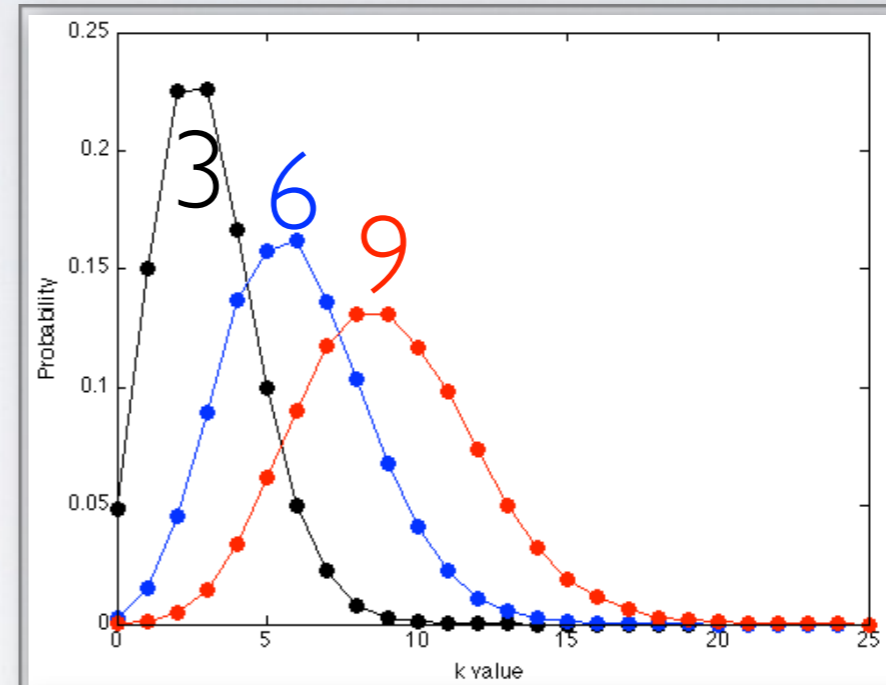
TIMING METHODS IN X-RAY ASTRONOMY

Tomaso Belloni (INAF - Osservatorio Astronomico di Brera)

POISSON NOISE EFFECTS

$$p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

- Counting detector
- Counting noise
- Background negligible
- Independent arrival times
- Exponential waiting time between photons



POWER SPECTRUM NORMALIZATION

- With this choice, noise power a χ^2 with 2 d.o.f.

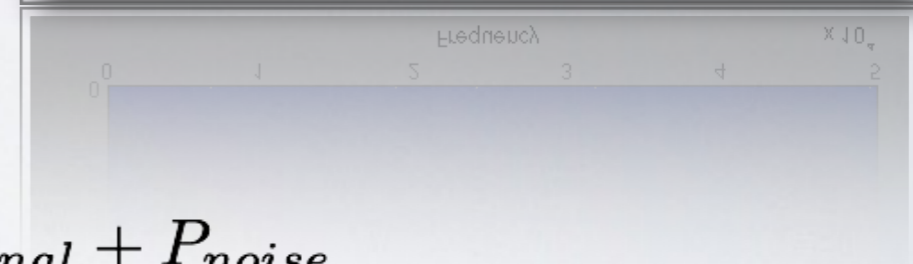
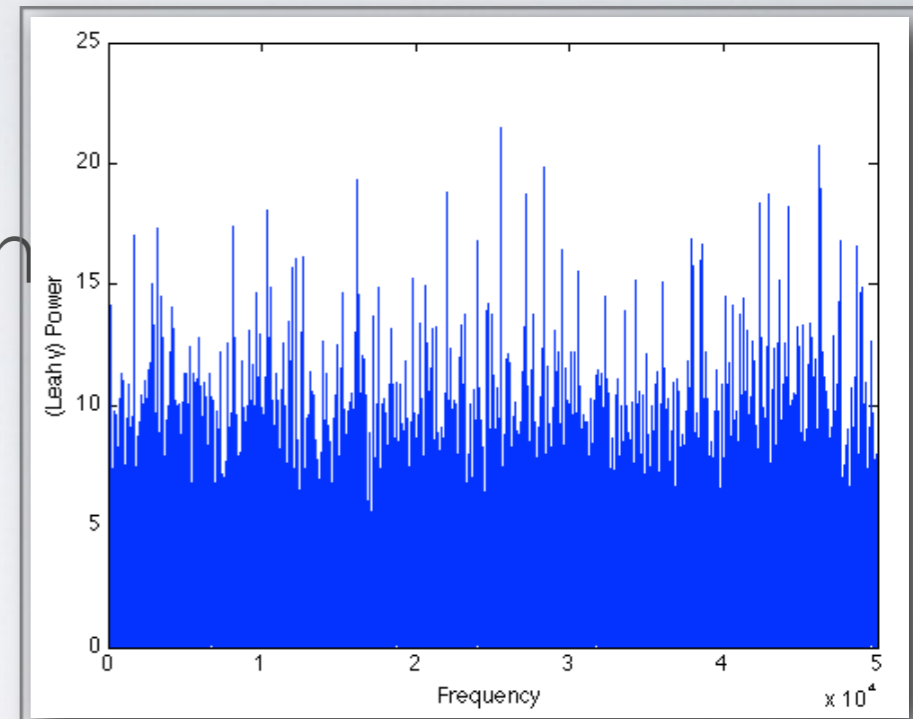
$$P = \frac{2}{N_{phot}} |a|^2$$

Leahy Norm.

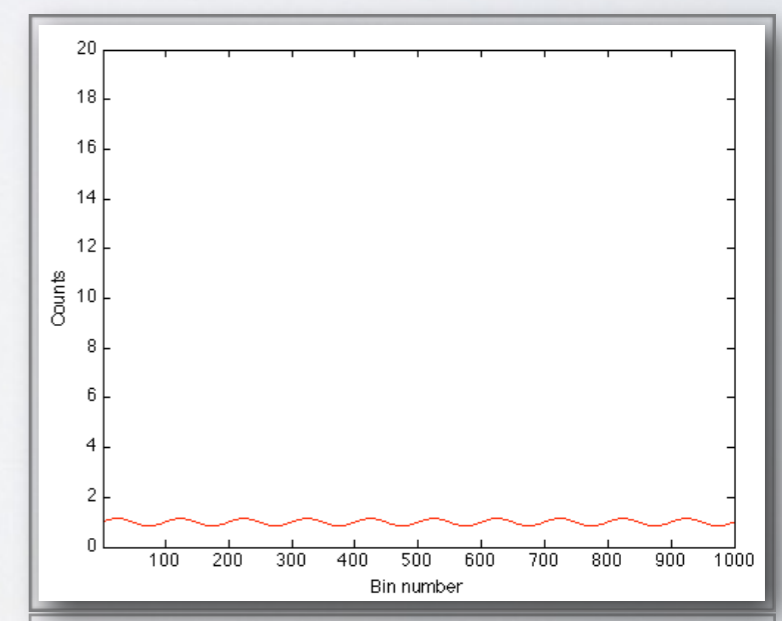
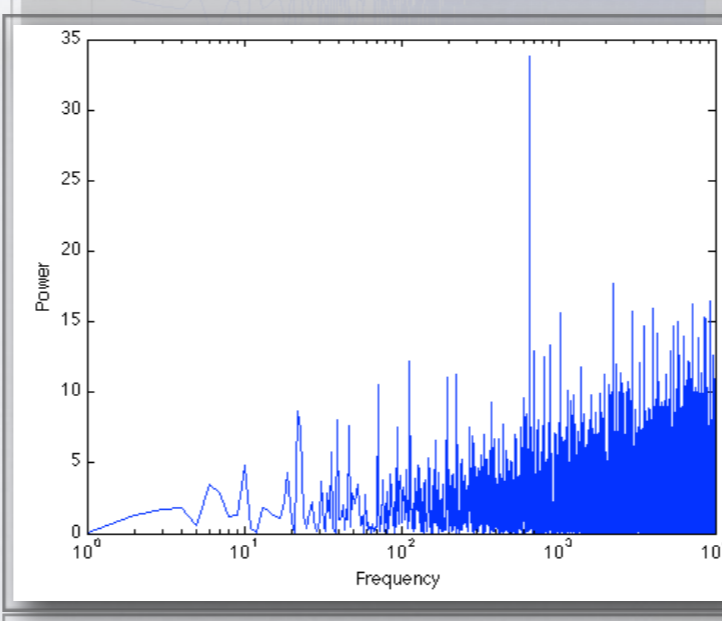
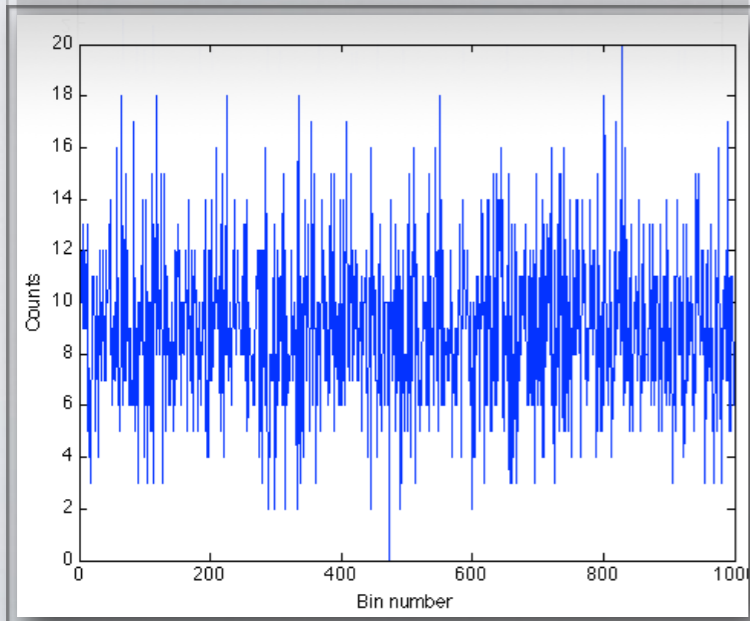
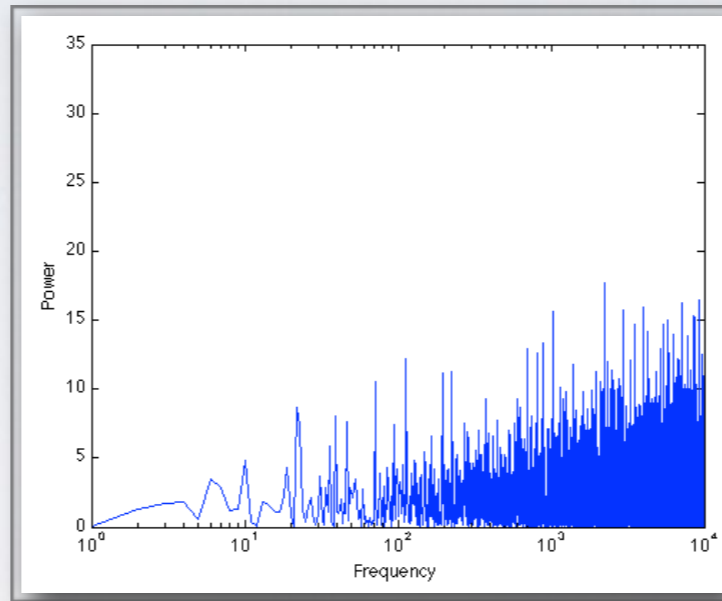
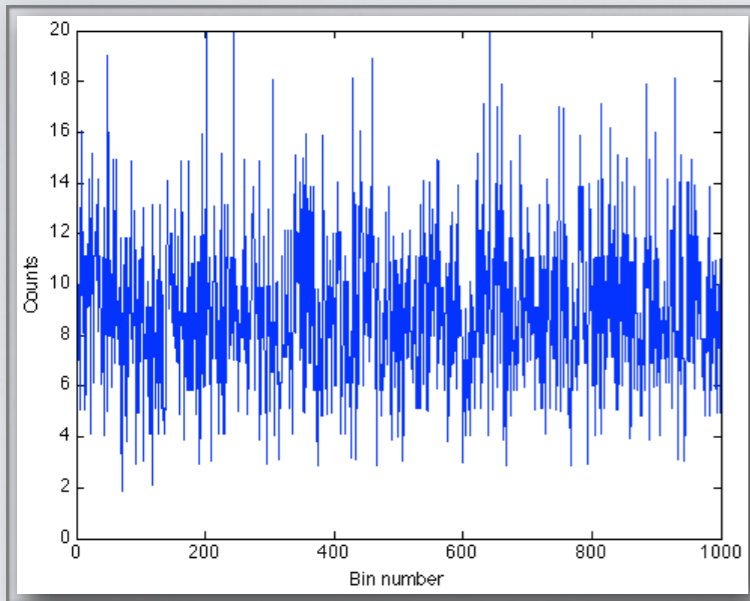
- Most noises do
- Average power is 2. I can calculate statistics

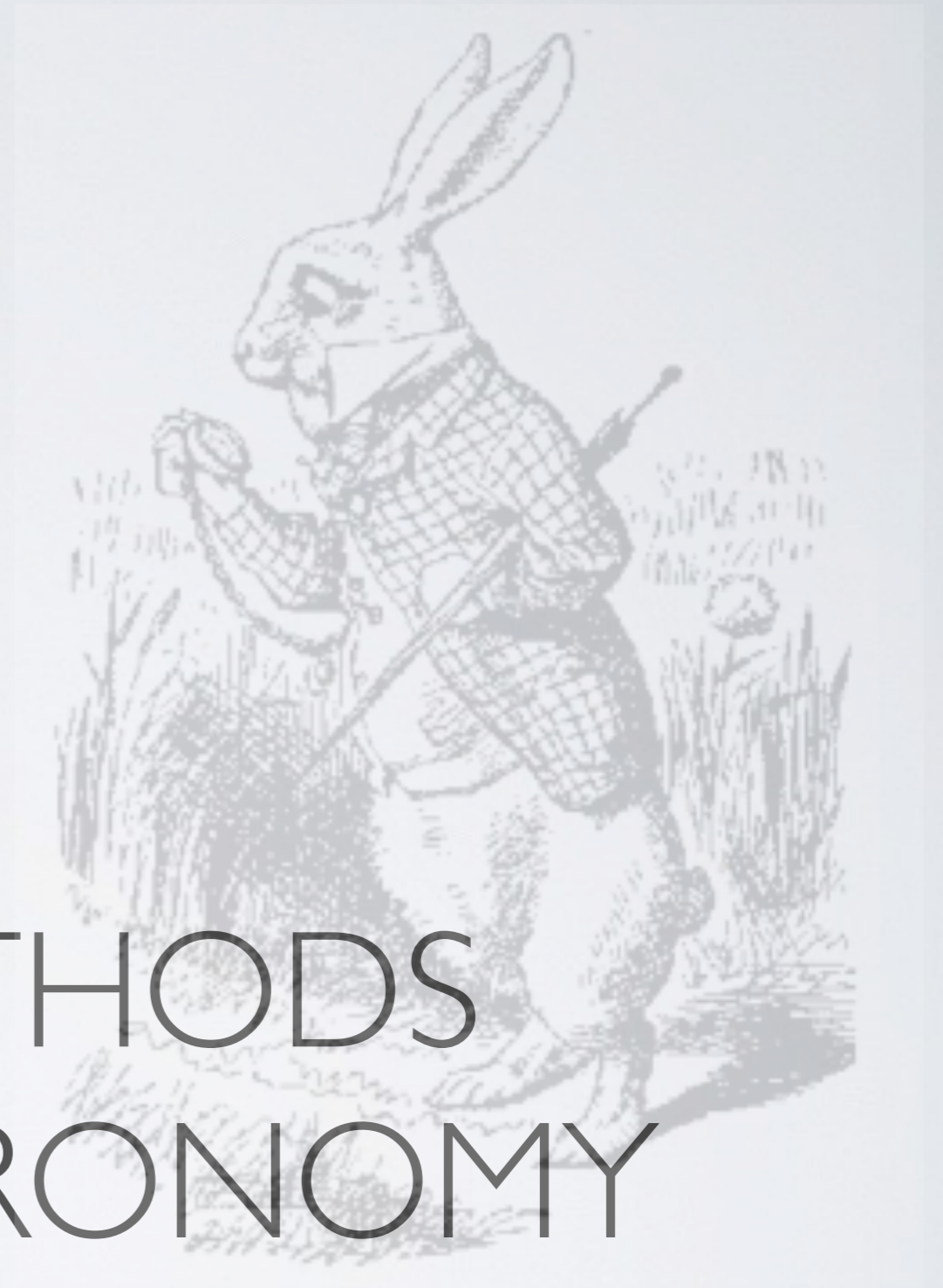
$$P_{total} = P_{signal} + P_{noise}$$

- Noise & signal independent:
- Not always so... (count rate!)



POWER OF POWER SPECTRUM



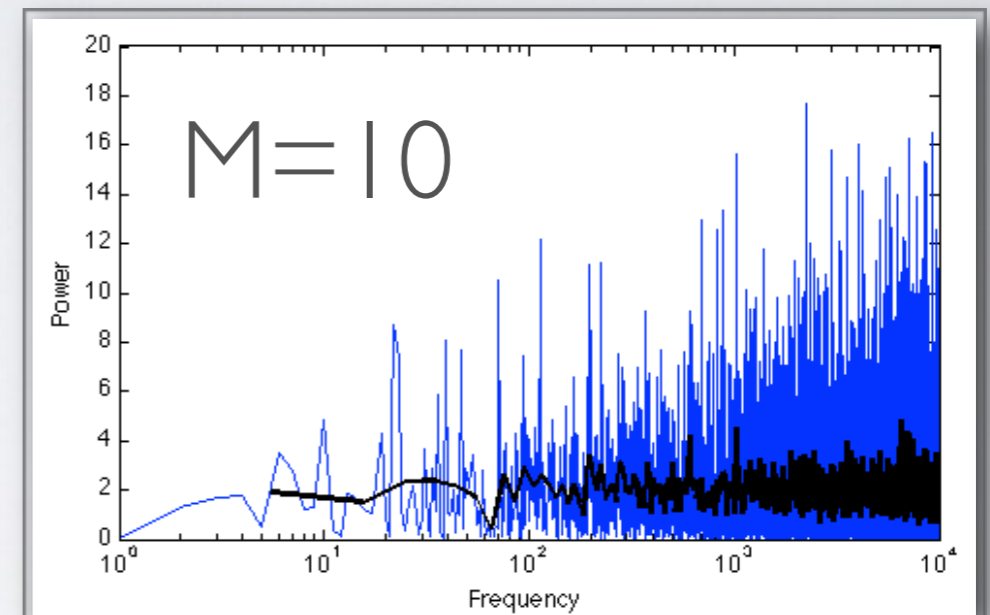


TIMING METHODS IN X-RAY ASTRONOMY

Tomaso Belloni (INAF - Osservatorio Astronomico di Brera)

NOISY NOISE

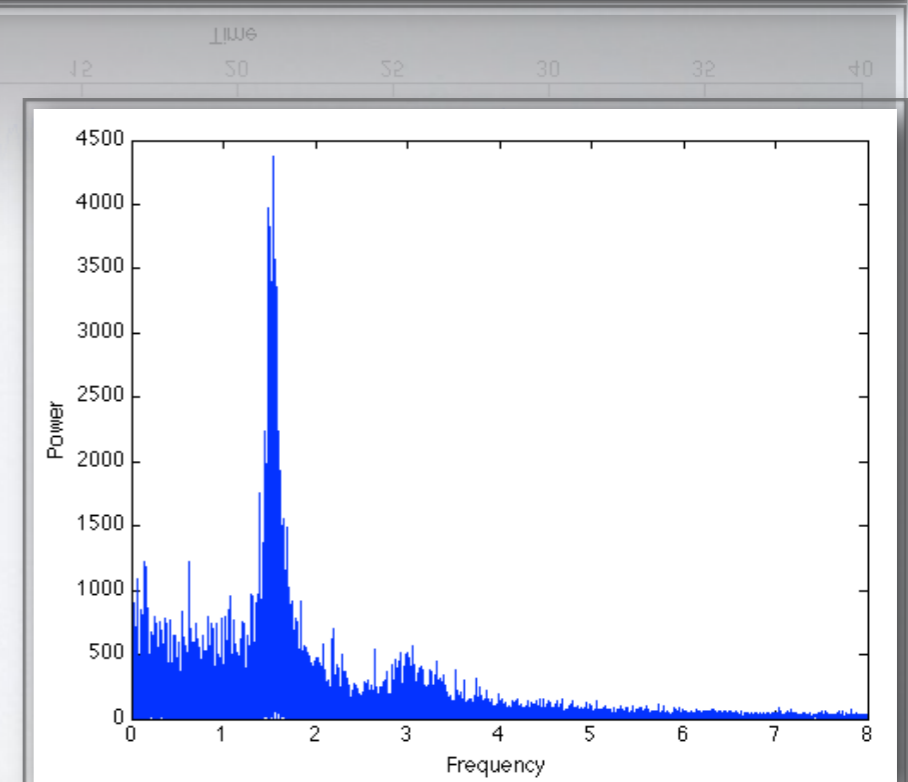
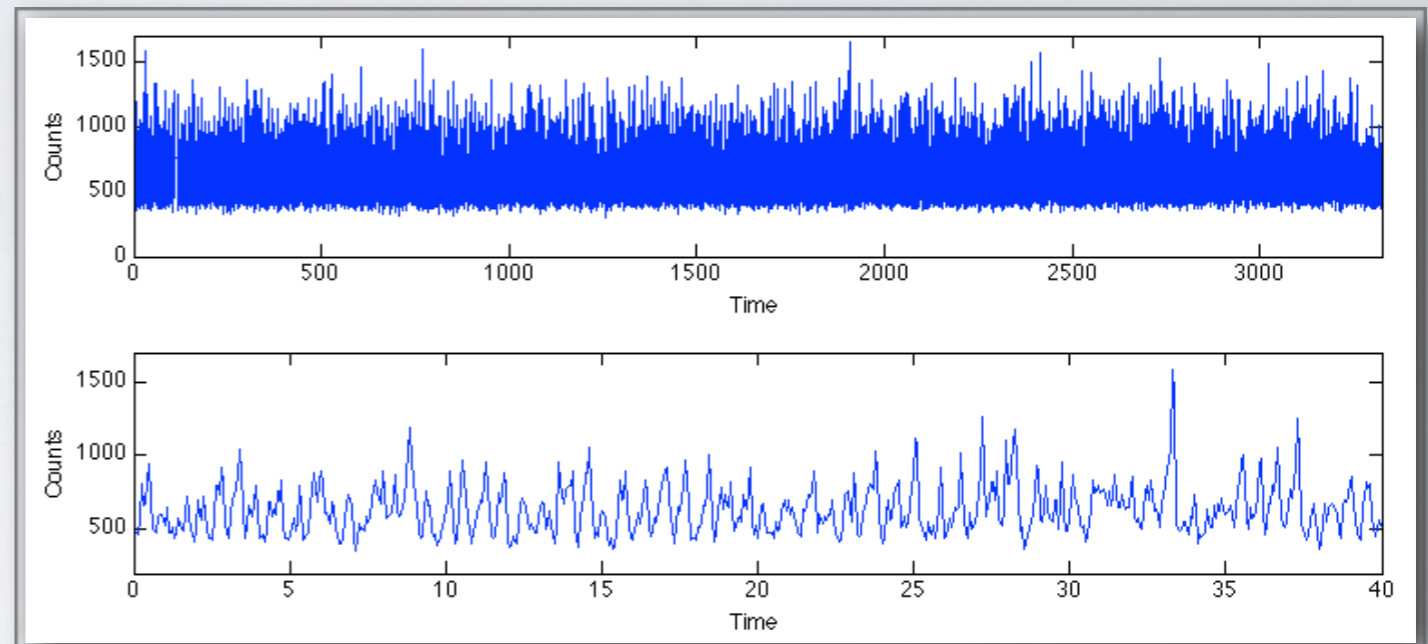
- ❖ Power spectrum of noise is very noisy! $\sigma_{P_j} = \langle P_j \rangle = 2$
- ❖ Increasing length or Δt not useful
- ❖ Two ways out:
 - ❖ a) Frequency rebinning by M
 - ❖ b) Time slicing by W and averaging powers



2 with 2MW dof
2MW distribution scaled by MW
Mean: 2
Standard dev: $\frac{2}{\sqrt{MW}}$

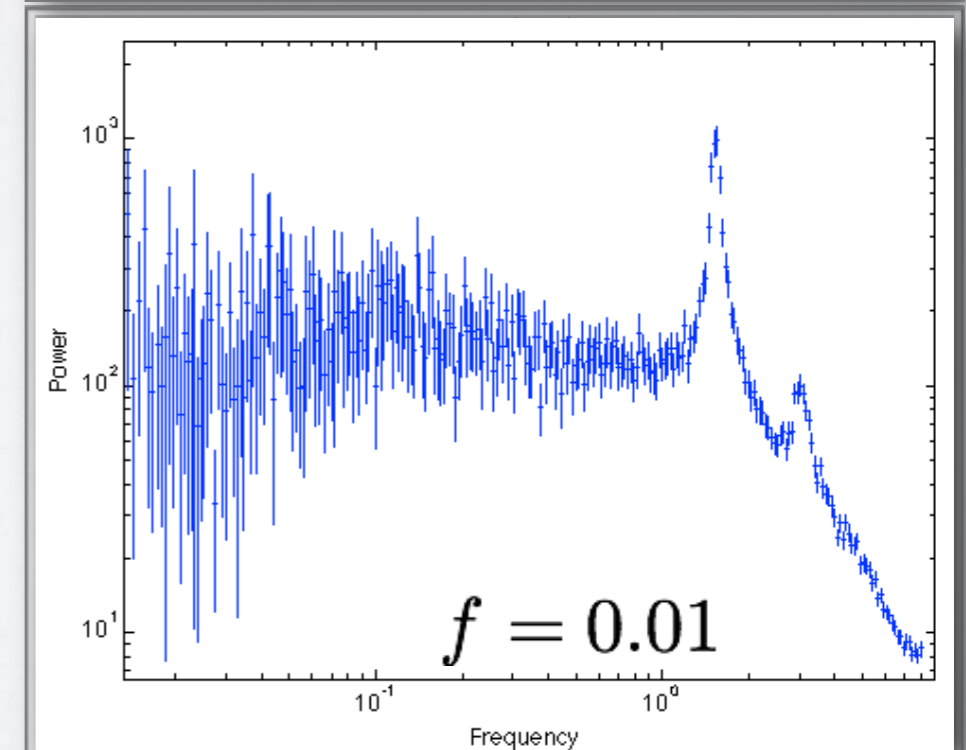
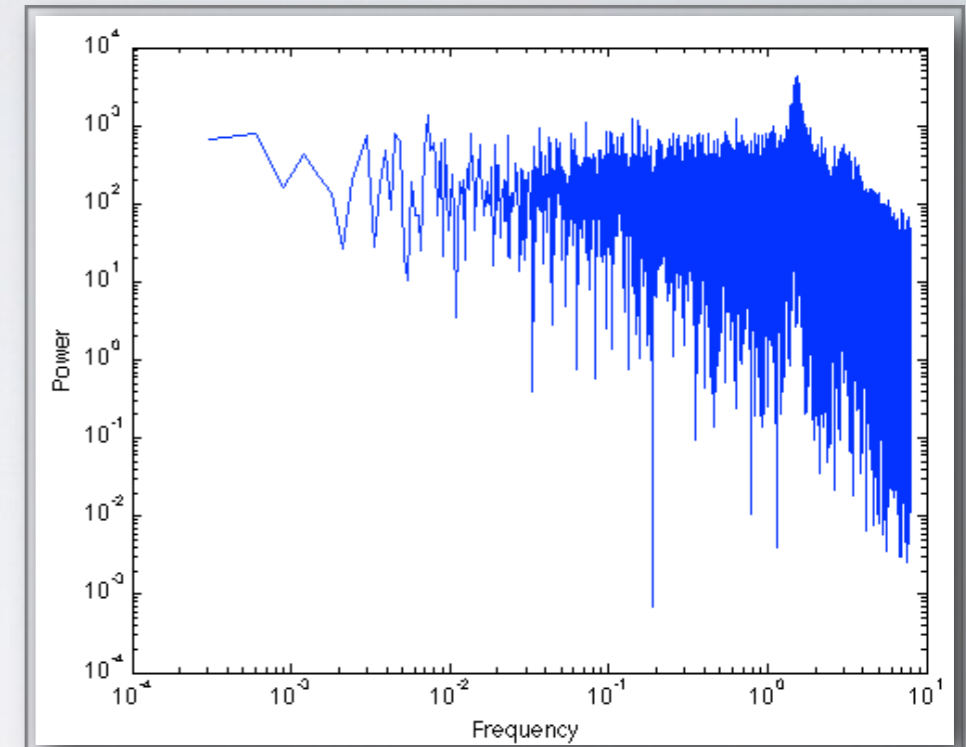
FULL POWER SPECTRUM

- ❖ RXTE light curve
- ❖ $t = 1/16$ seconds
- ❖ $T = 3325$ seconds
- ❖ Something can be seen by eye in the light curve
- ❖ Full power spectrum
- ❖ High-power signal, no coherent peak



LOG SPACE AND REBINNING

- ❖ Log-log plot more appropriate for all frequencies
- ❖ Errors are 100%
- ❖ Frequency rebinning (M)
- ❖ Log-rebinning: $\Delta\nu_j = \Delta\nu_{j-1} * (1 + f)$
- ❖ Error bars, better shape
- ❖ Poisson level below scale

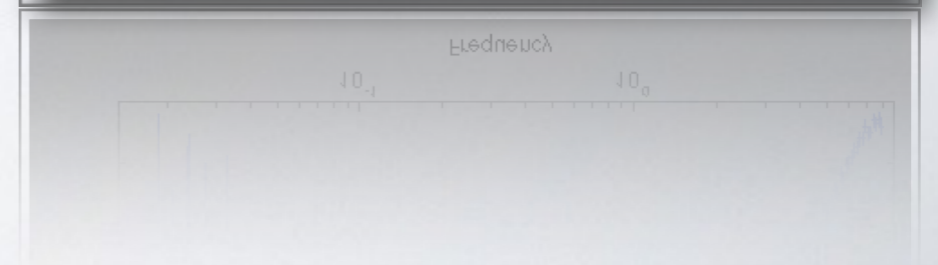
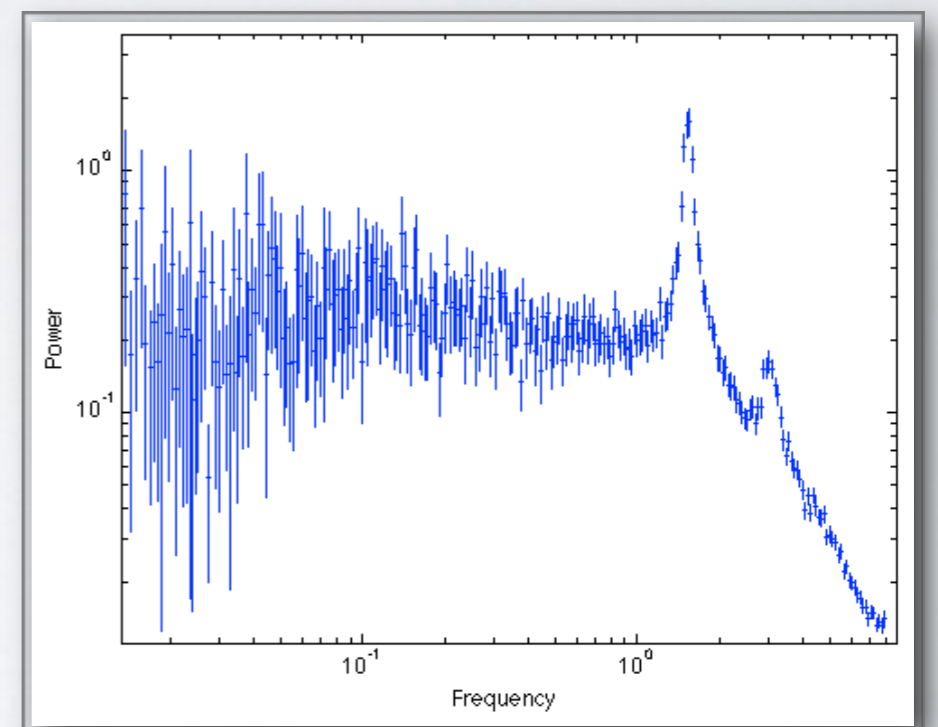


NORMALIZATION

- ❖ Leahy normalization very useful for statistics
- ❖ Power \propto square intensity
- ❖ Remove it by dividing by square intensity: rms (Belloni) normalization
- ❖ Caveat: from Leahy to rms²
- ❖ Meaning: squared rms per decade
- ❖ Root of integral gives fractional rms

$$P_{rms} = \frac{P}{C^2}$$

$$P_{rms} = \frac{P_{Leahy}}{C}$$

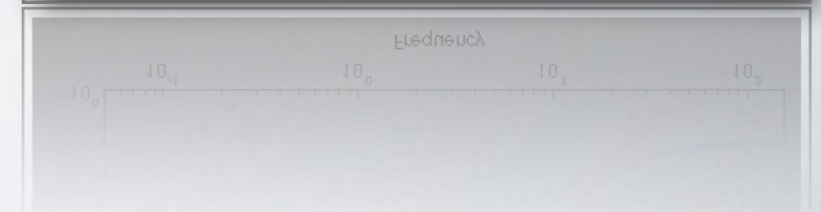
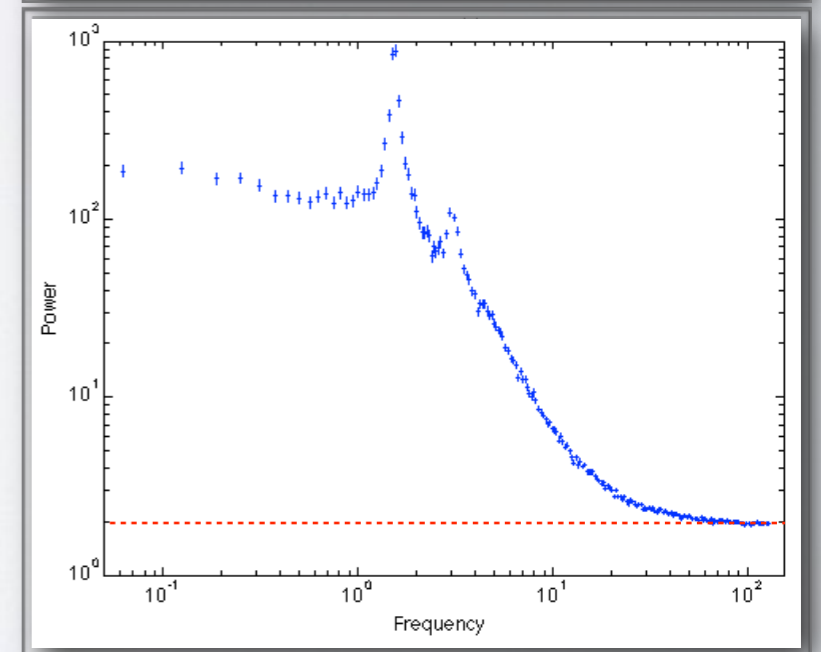
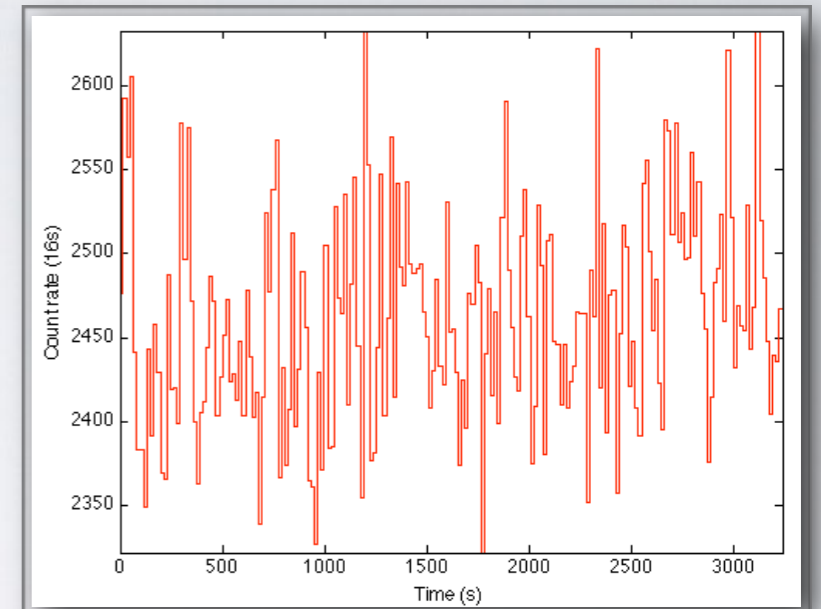


A NOTE ABOUT REBINNING

- ❖ Coherent peak: narrow power distribution - least rebinning - the longer the observation span, the better
- ❖ Broad peak: broad power distribution - rebinning helps - length of observation not crucial
- ❖ Very important for maximizing sensitivity

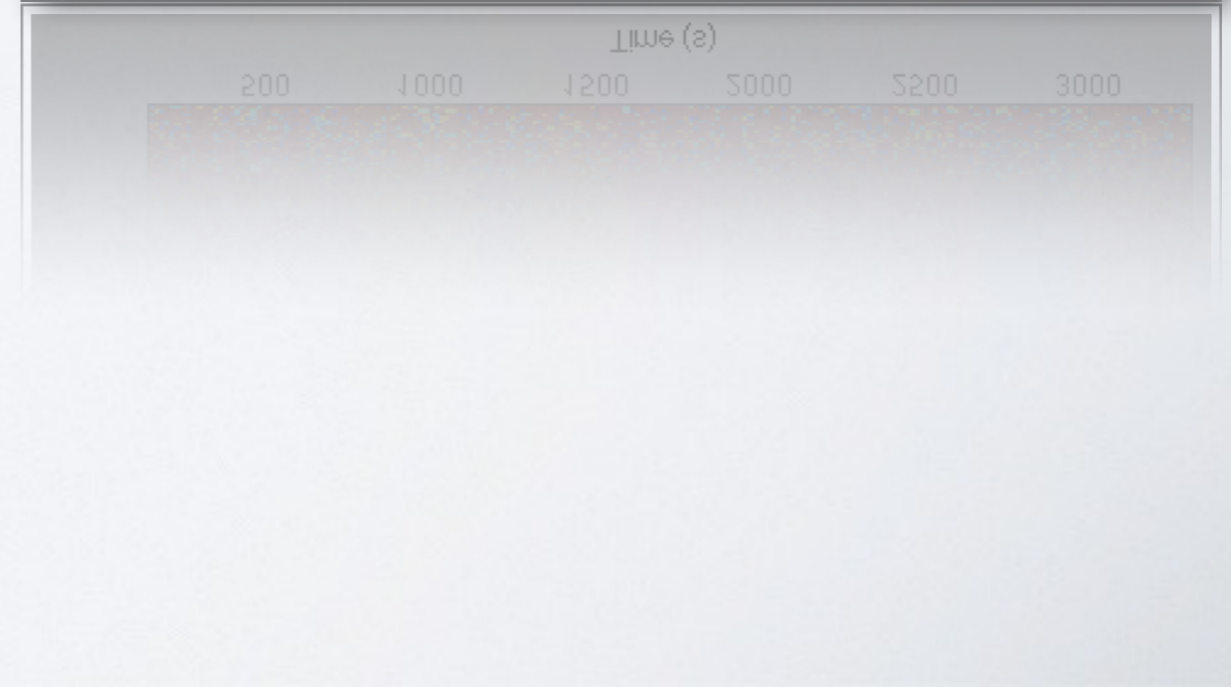
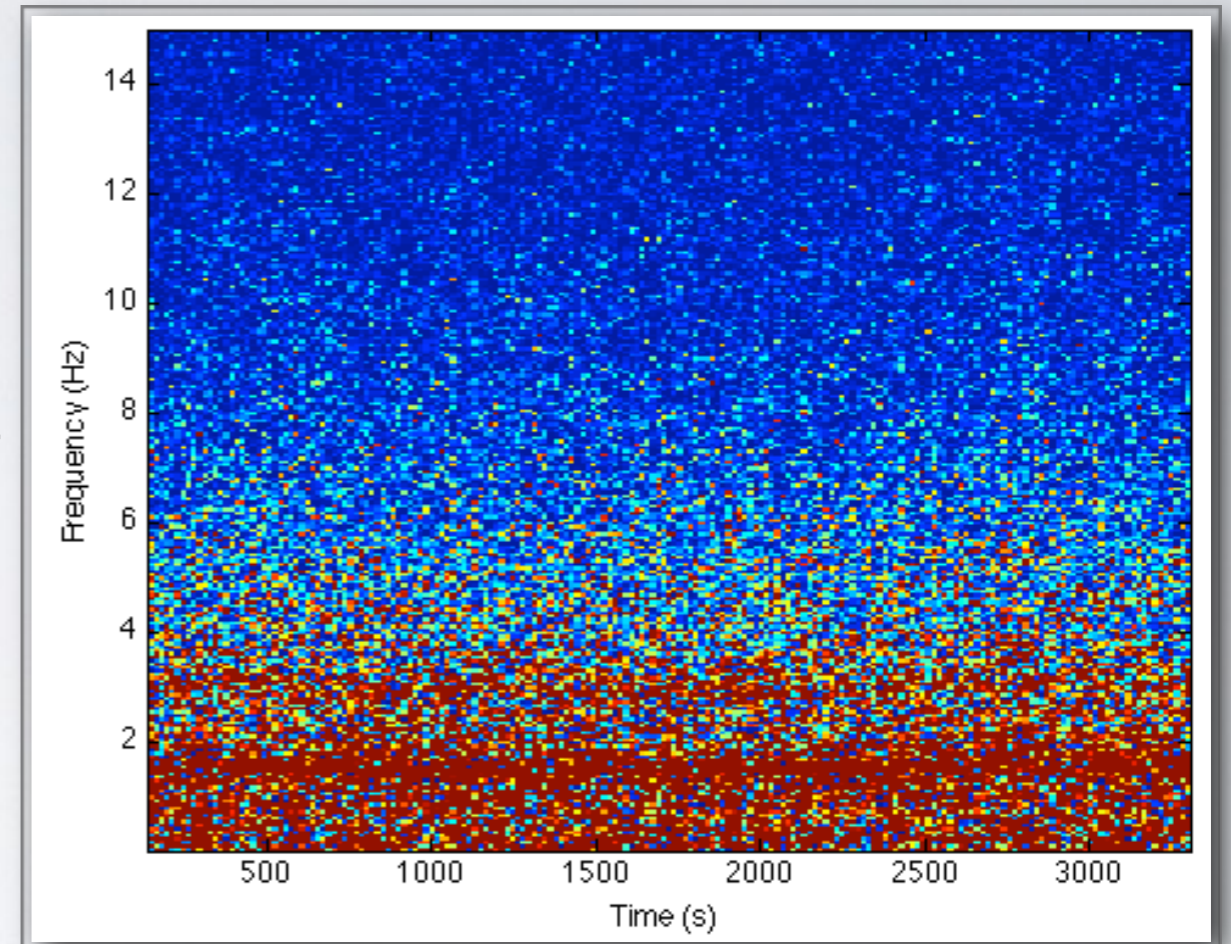
W: WELCH POWER SPECTRUM

- ❖ If signal **stationary**
- ❖ Slice the signal
- ❖ Power spectrum of slices
- ❖ Add the W slices
- ❖ Sliding slices are also possible (statistics?)
- ❖ Windowing is also possible



TIME-FREQUENCY ANALYSIS

- ❖ If signal is not *stationary*
- ❖ No average of power spectra
- ❖ Image: time-frequency-power
- ❖ Uncertainty principle



THE UNCERTAINTY PRINCIPLE

$$T^2 = \sigma_t^2 = \int (t - \langle t \rangle)^2 |s(t)|^2 dt$$

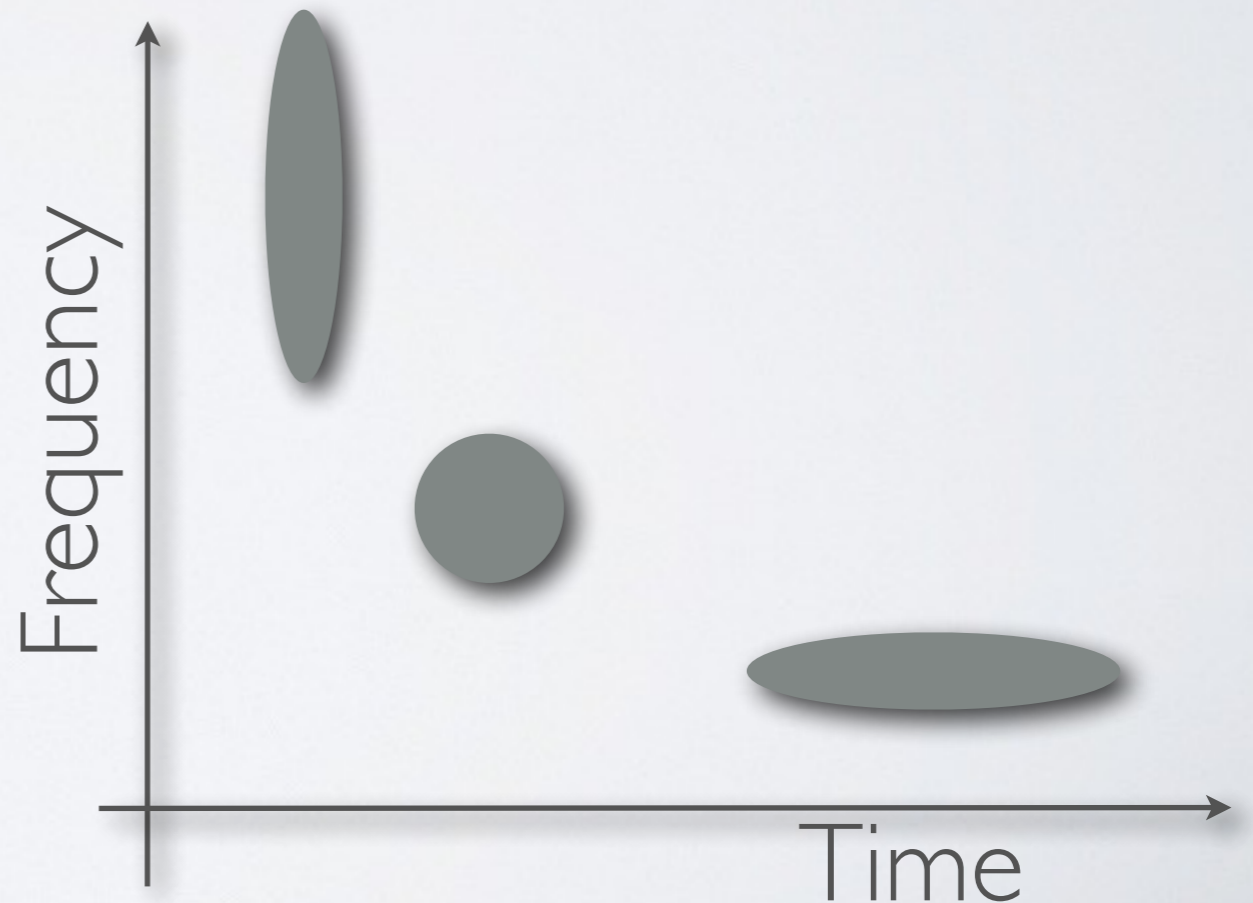
$$B^2 = \sigma_\omega^2 = \int (\omega - \langle \omega \rangle)^2 |S(\omega)|^2 d\omega$$

$$TB \geq \frac{1}{2}$$

σ_t Duration

σ_ω Bandwidth

- You cannot beat it
- It's a big limitation



THE EASY WAY OUT

- Spectrogram (from short-term Fourier Transform)
- Sliding window to select time (window can be chosen)
- Obtain a time-frequency image

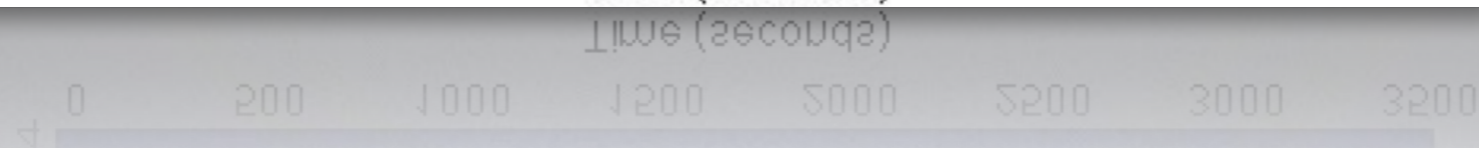
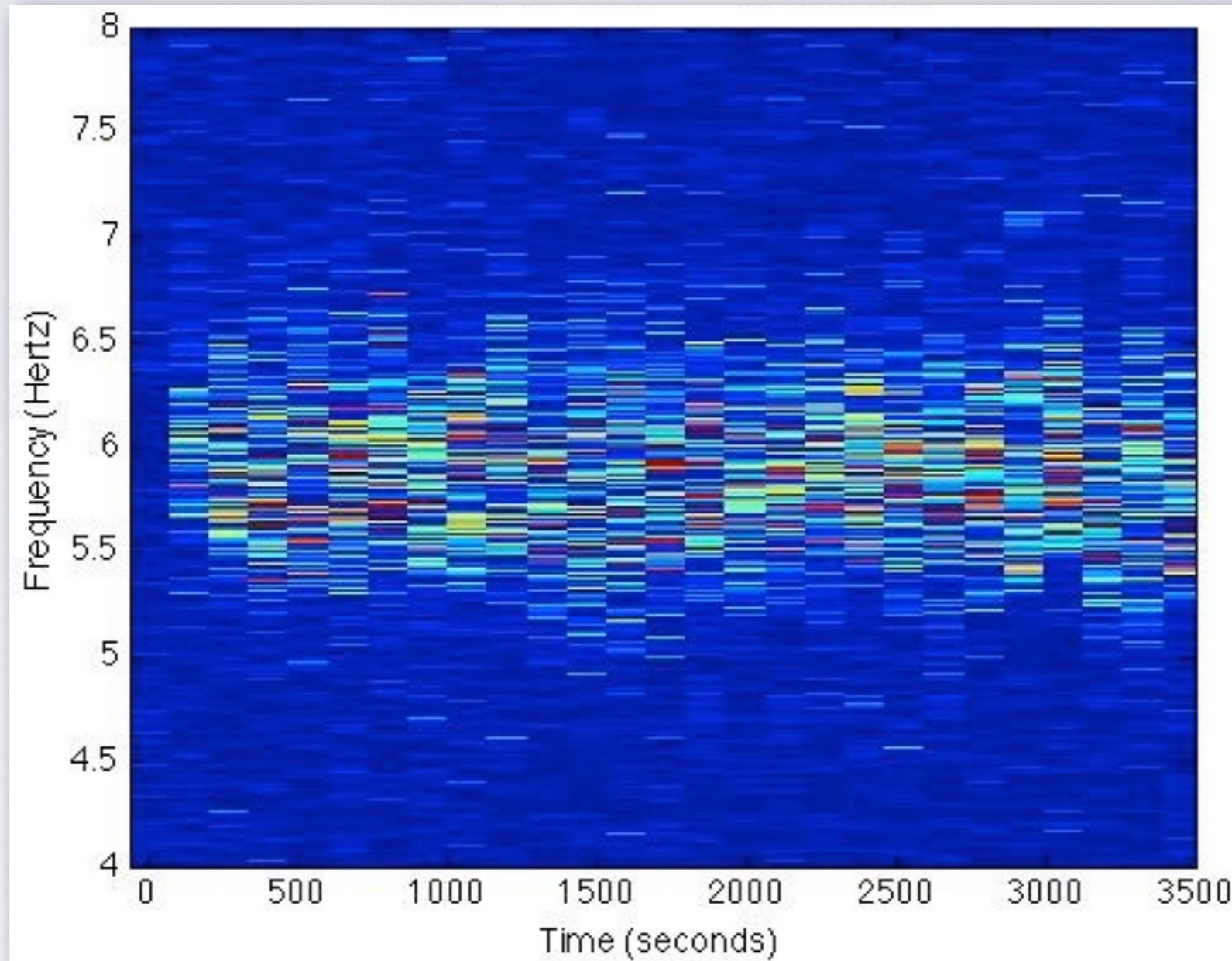
$$s_t(\tau) = s(\tau)h(\tau - t)$$

$$P(t, \omega) = \left| \frac{1}{\sqrt{(2\pi)}} \int e^{(-i\omega\tau)} s(\tau)h(\tau - t)d\tau \right|^2$$

AN EXAMPLE

- Quasi-Periodic Oscillation

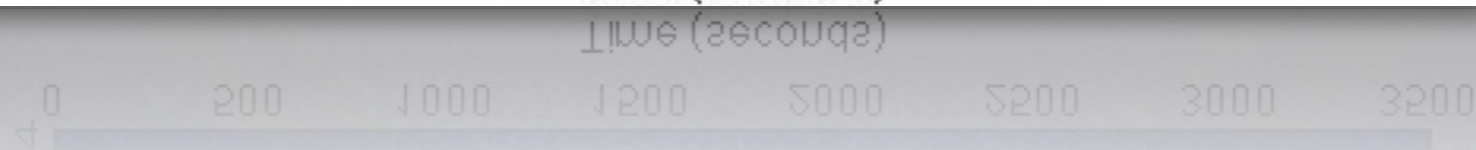
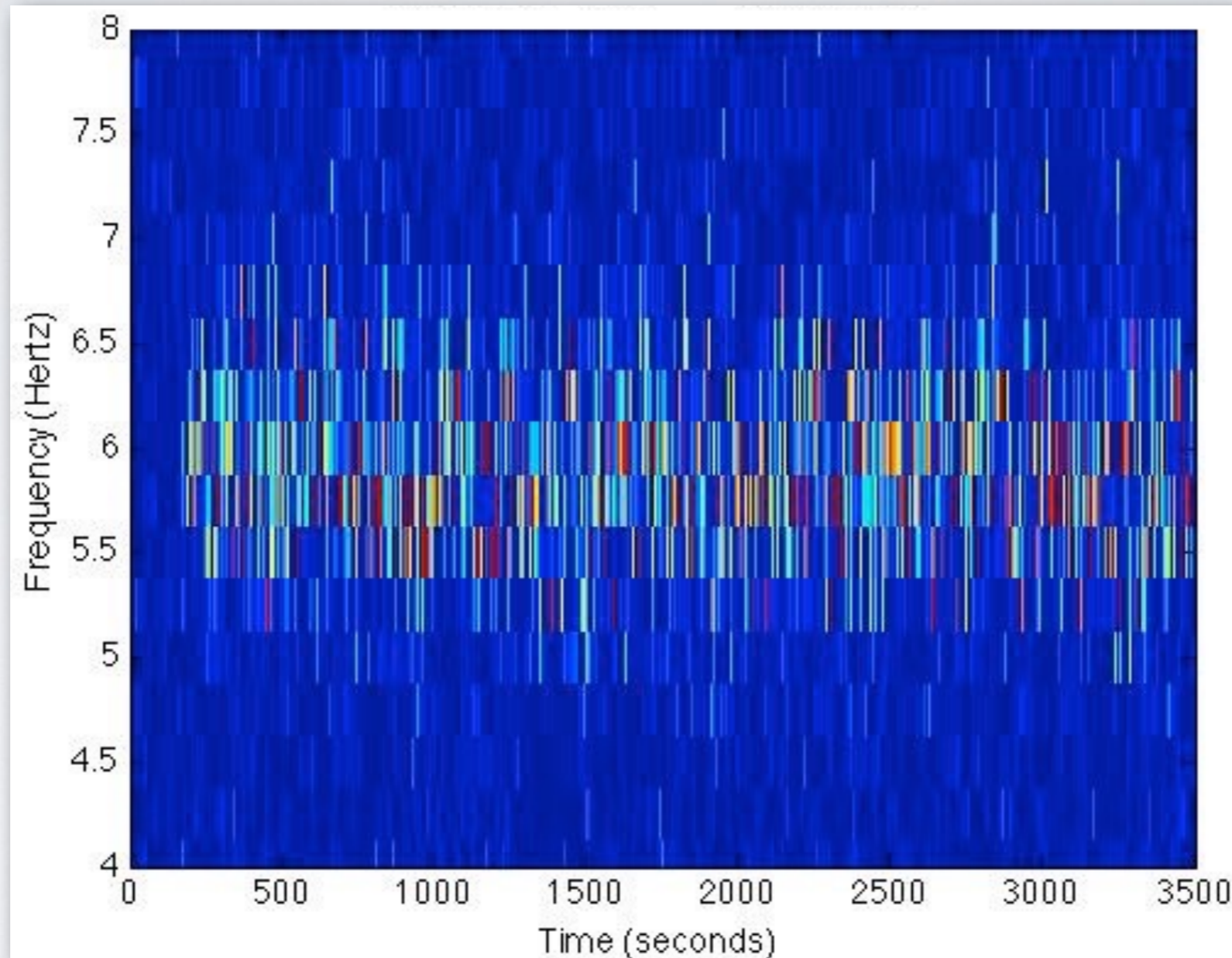
$$T = 128 \text{ s}$$



AN EXAMPLE

- Quasi-Periodic Oscillation

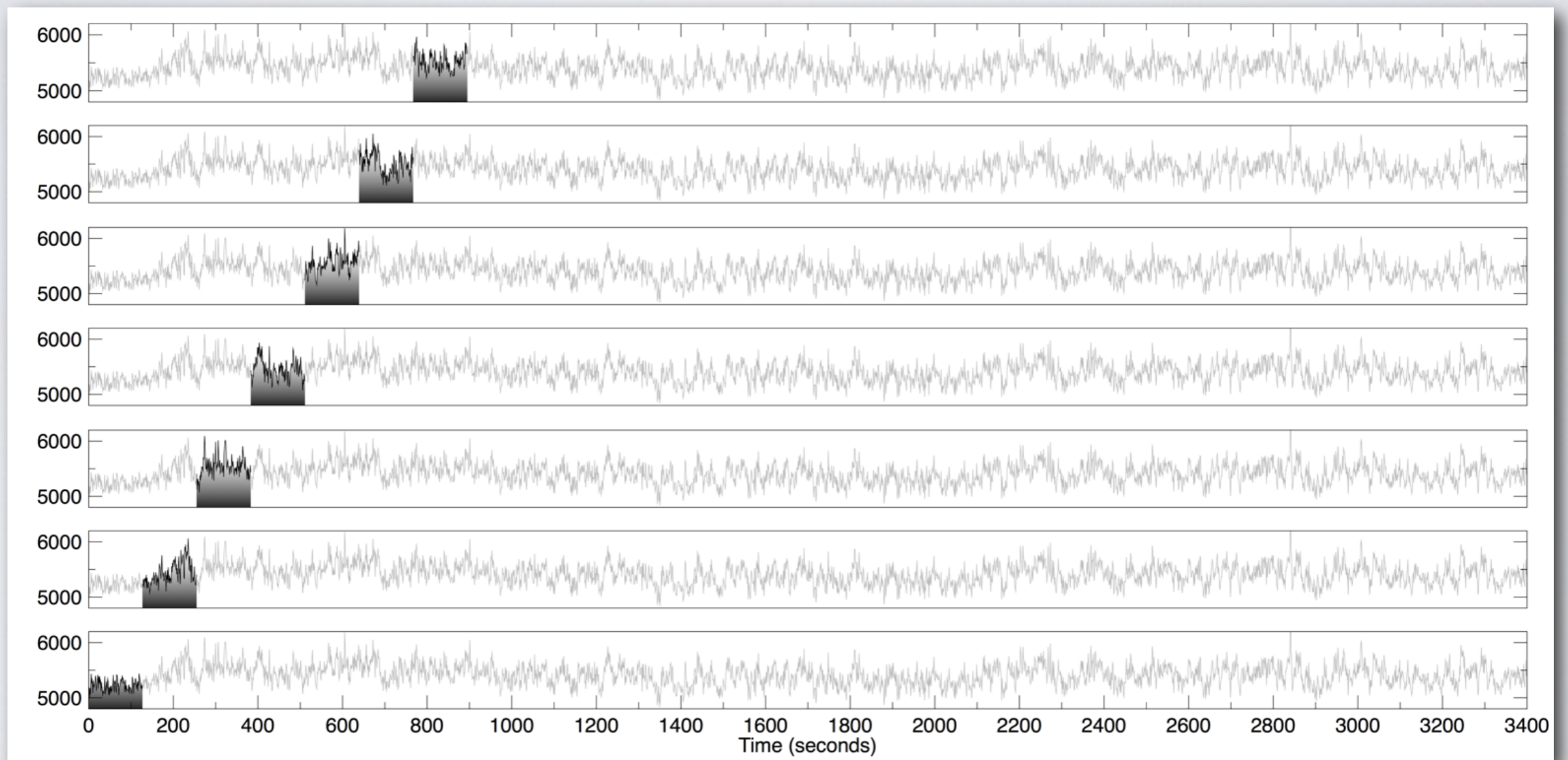
$$T = 4 \text{ s}$$



NON-OVERLAPPING

- Sliding window to select time $s_t(\tau) = s(\tau)h(\tau - t)$

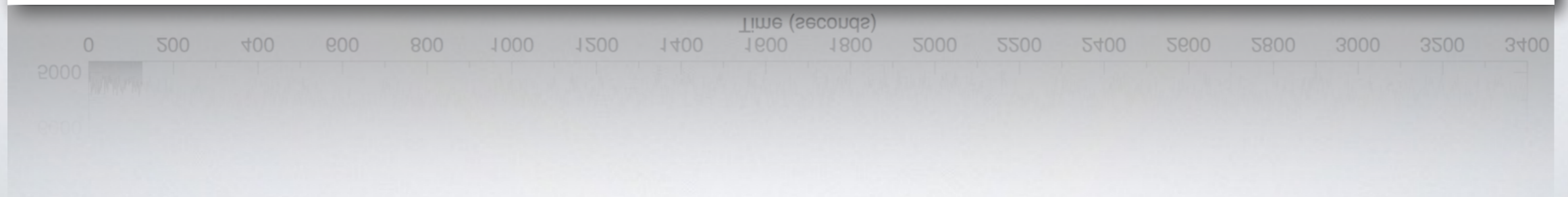
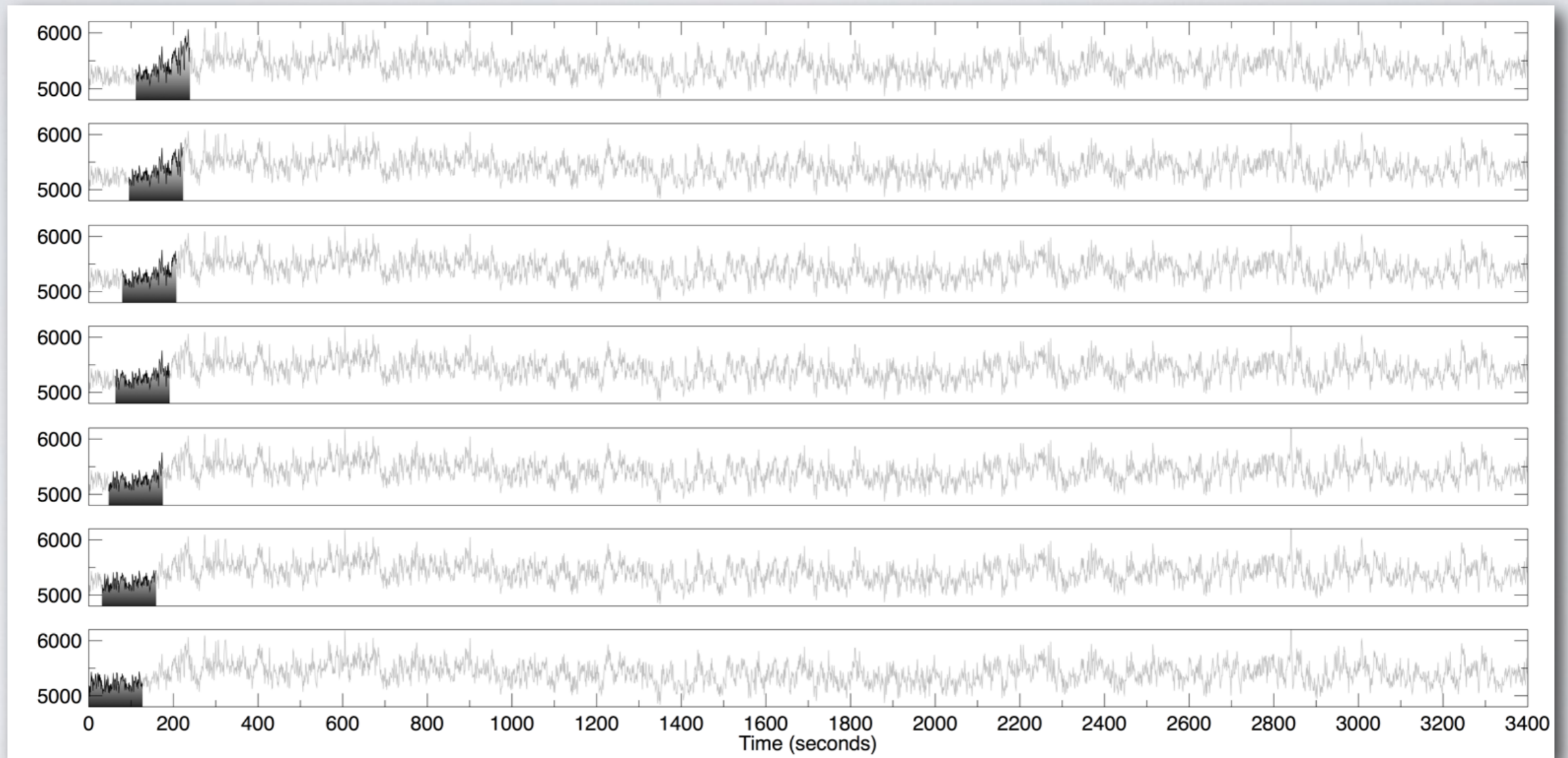
$$t = \tau$$



OVERLAPPING

- Sliding window to select time $s_t(\tau) = s(\tau)h(\tau - t)$

$$t < \tau$$

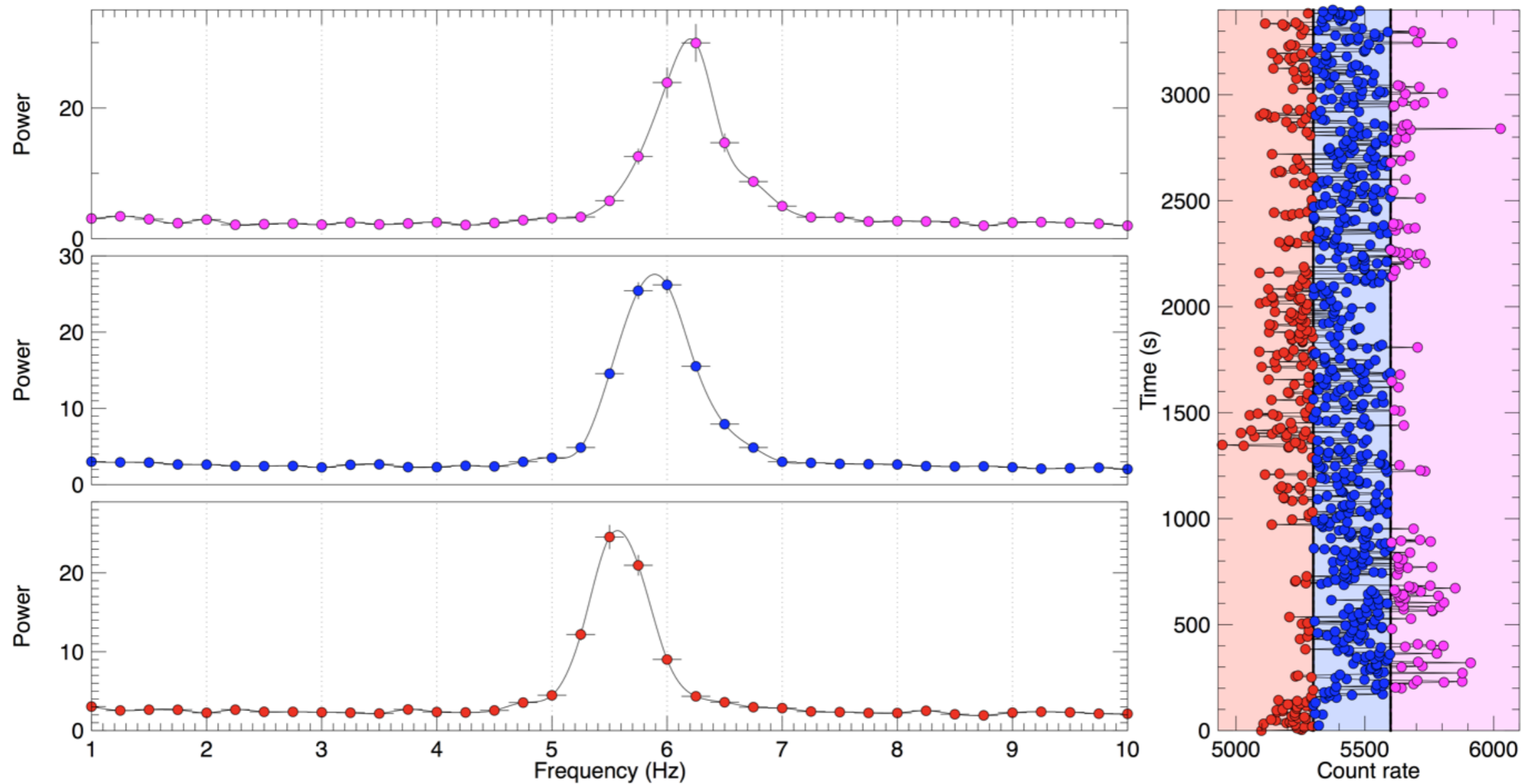


SHIFT 'N' ADD TECHNIQUE

- ❖ Used for twin high-frequency peaks
- ❖ You see one, not the other
- ❖ The one you see moves
- ❖ Correct for the movement, align the spectra in an additive way
- ❖ More complex: multiplicative technique (tricky to implement)

LINEAR SHIFT AND ADD

- Good to recover features at a constant distance in ν



INSTRUMENTAL DEAD TIME

- ❖ After a photon, dead time
- ❖ Introduces correlations between photons (no Poisson!)
- ❖ It must be as small as possible and well-known and modeled
- ❖ Two types of dead time:
 - ❖ Paralyzable
Every incident event causes a dead time t_d even if it's not detected
 - ❖ Non-paralyzable
Only a detected event causes a dead time t_d

PARALYZABLE DEAD TIME

❖ If incident rate r_{in} is very high, no detected counts at all!

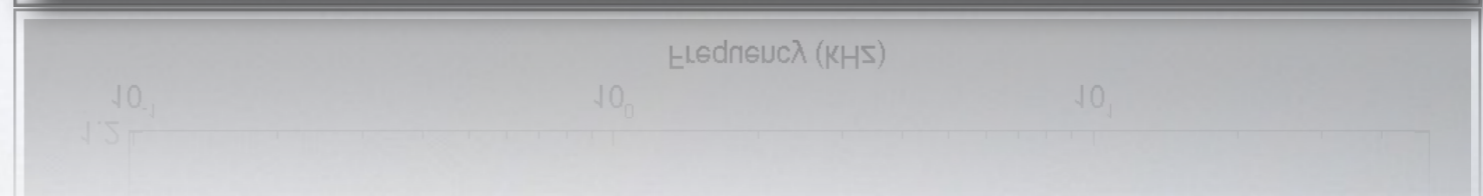
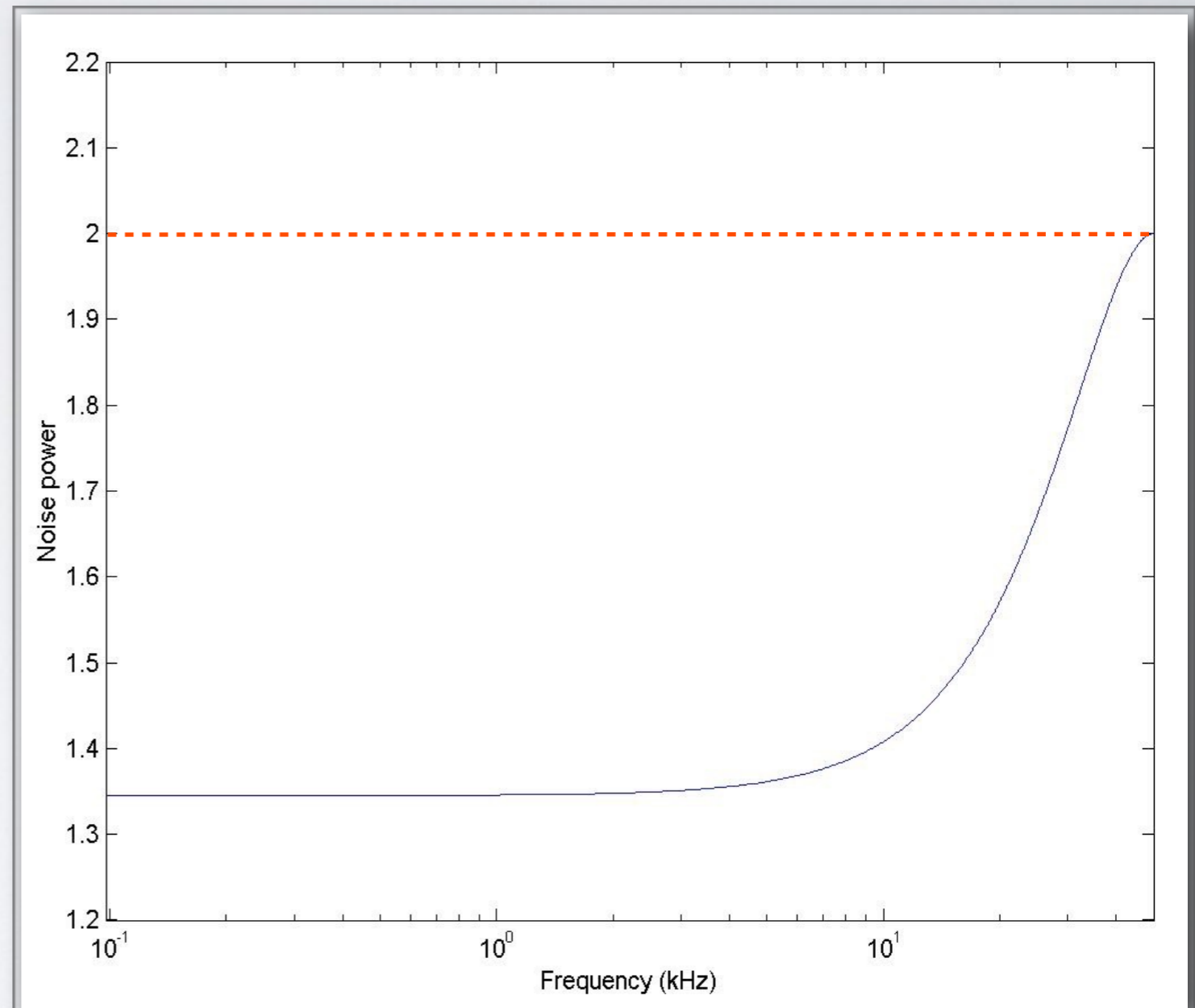
❖ Detected rate: $r_0 = r_{in} e^{-r_{in} t_d}$ $\lim_{r_{in} \rightarrow +\infty} r_0 = 0$

❖ In RXTE/PCA, for binning time $t_b \geq t_d$

$$\langle P_j \rangle = 2 \times \left[1 - 2r_0 t_d \left(1 - \frac{t_d}{2t_b} \right) \right] - 2 \frac{N-1}{N} r_0 t_d \left(\frac{t_d}{t_b} \right) \cos \left(\frac{2\pi j}{N} \right)$$

PARALYZABLE DEAD TIME

$r_{in} = 20$ kcts/s
 $r_0 = 16.385$ kcts/s
 $t_d = t_b = 10$ μ s
 $N = 1024$

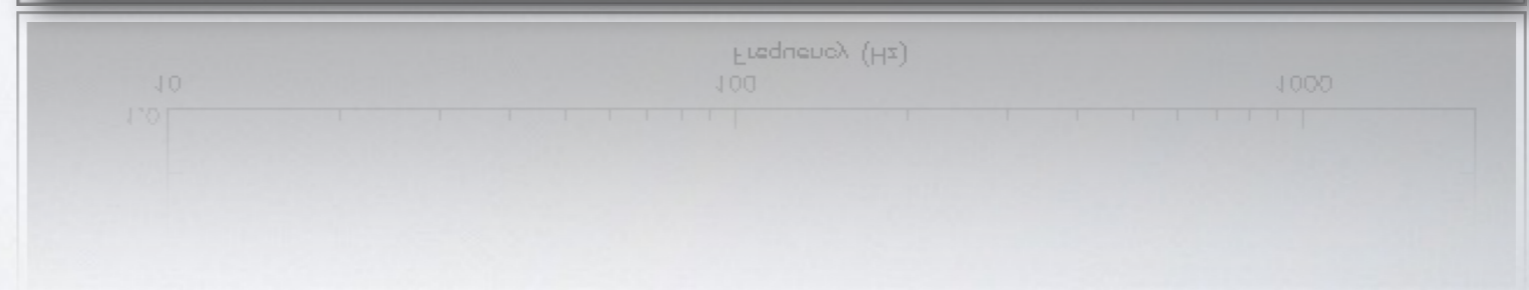
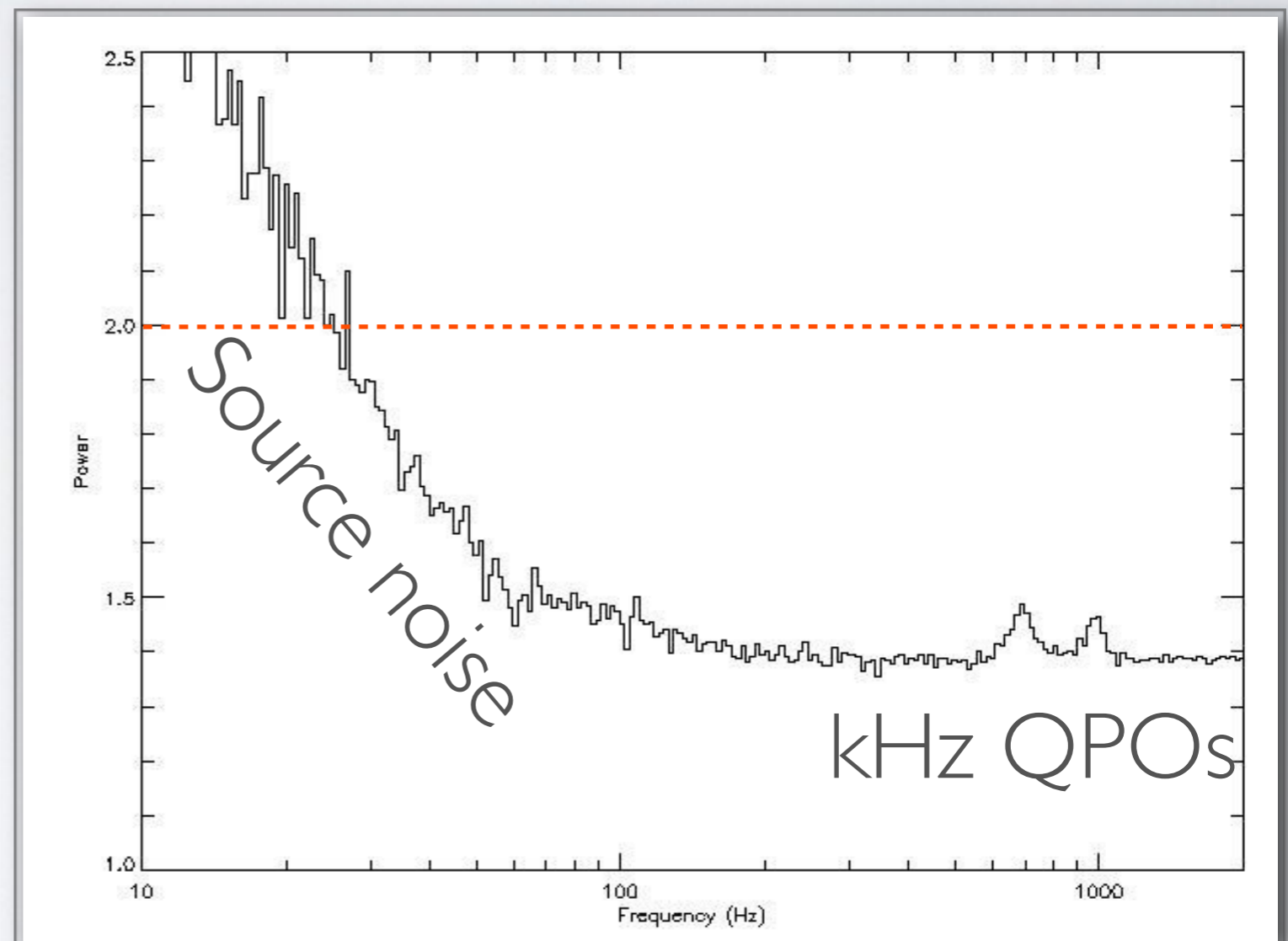


NON-PARALYZABLE DEAD TIME

- ✦ If incident rate r_{in} is very high, one count every t_d
- ✦ Detected rate: $r_0 = \frac{r_{in}}{1 + r_{in}t_d}$ $\lim_{r_{in} \rightarrow +\infty} r_0 = t_d^{-1}$
- ✦ Formula is even more complicated, result is similar
 - ✦ Depression of noise level @ low frequencies (correlation)
 - ✦ Peak @ t_d (quasi-periodicity)

PARALYZABLE DEAD TIME: SCO X-1

$$r_0 = 10^5 \text{ cts/s}$$
$$t_d = 10 \mu\text{s}$$



FITTING POWER SPECTRA

- ❖ Fit with typical minimization (χ^2)
- ❖ Rebinning is important for χ^2
- ❖ Error estimation vs. significance
- ❖ Limit in power and **NOT** rms

- ❖ Coherent peaks: distribution of powers and number of trials

NUMBER OF TRIALS

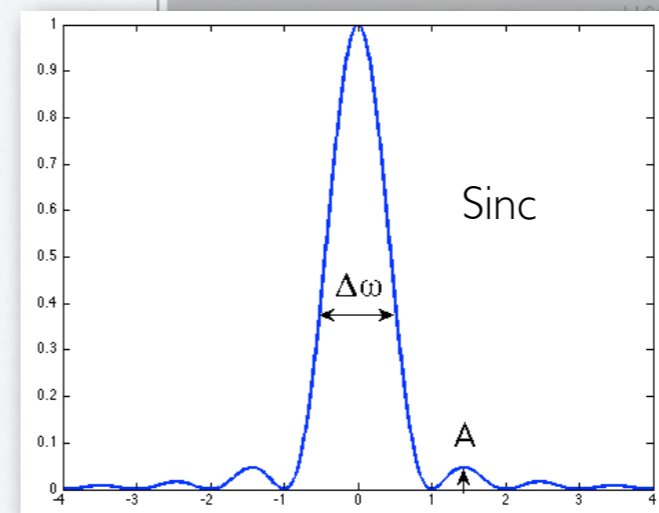
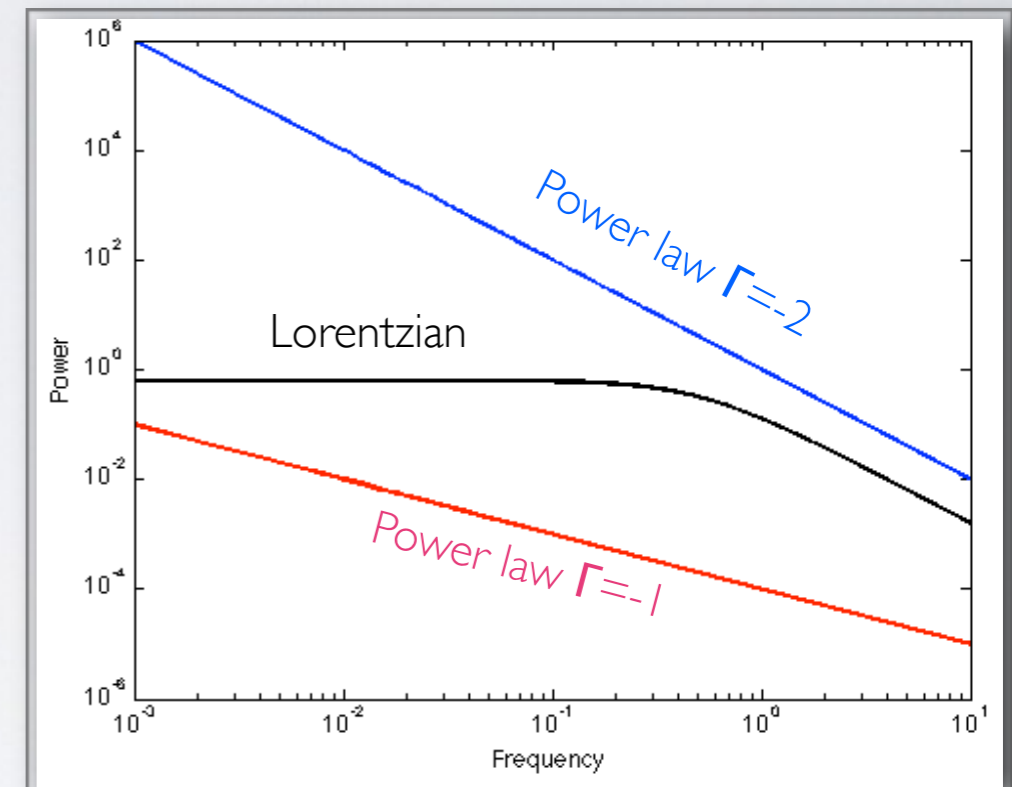
- ❖ Important statistical concept
- ❖ Should be done correctly, but if P is small can be approximated

$$\tilde{P}_{chance} = P_{chance} \times N_{trials}$$

- ❖ IMPORTANT: how to estimate N_{trials}
- ❖ For Power Spectra: number of *independent* frequencies

CONTINUUM COMPONENTS

- ❖ Very important for accreting sources
- ❖ Slope is limited by the window
- ❖ Window overflow
- ❖ $\Gamma = -2$ is the steepest value
- ❖ If an issue (pulsar noise):
exotic methods

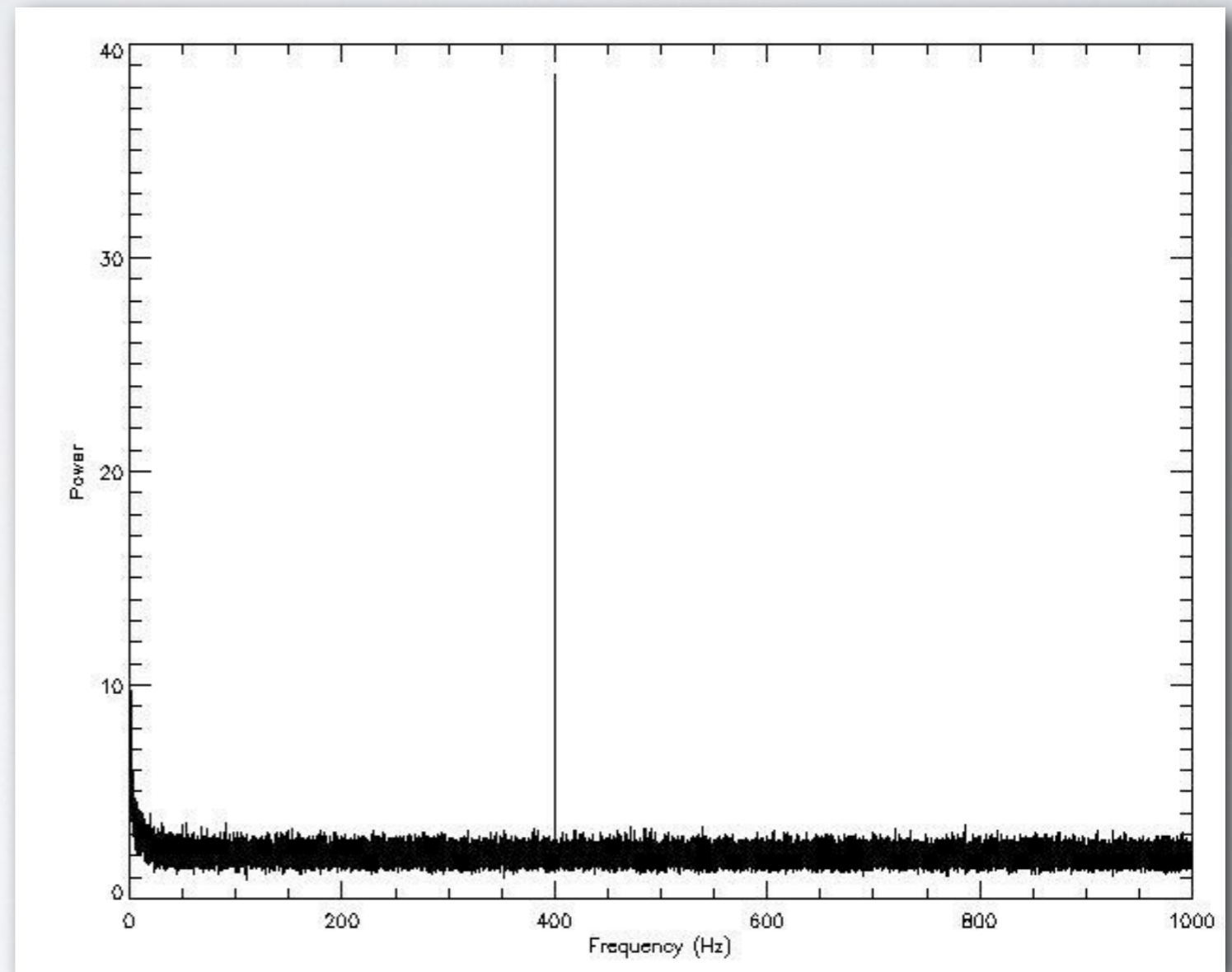


MAINTYPES OF SIGNALS

- ❖ Coherent pulsation
- ❖ Broad-band noise
- ❖ Broad peak (QPO)
- ❖ “Peaked-noise”

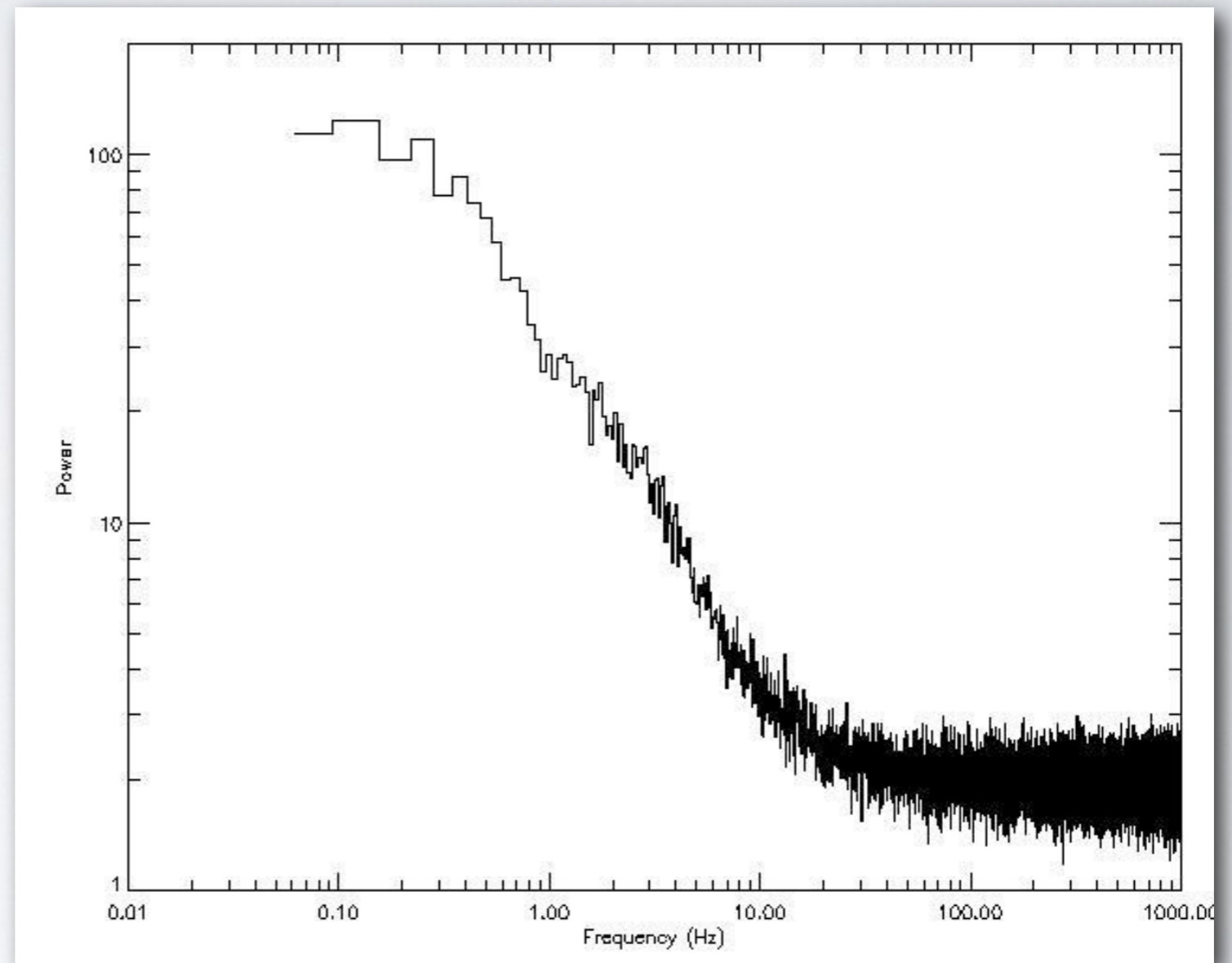
MAIN TYPES OF SIGNALS

- ❖ Coherent pulsation
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- ❖ Broad peak (QPO)
- ❖ “Peaked-noise”



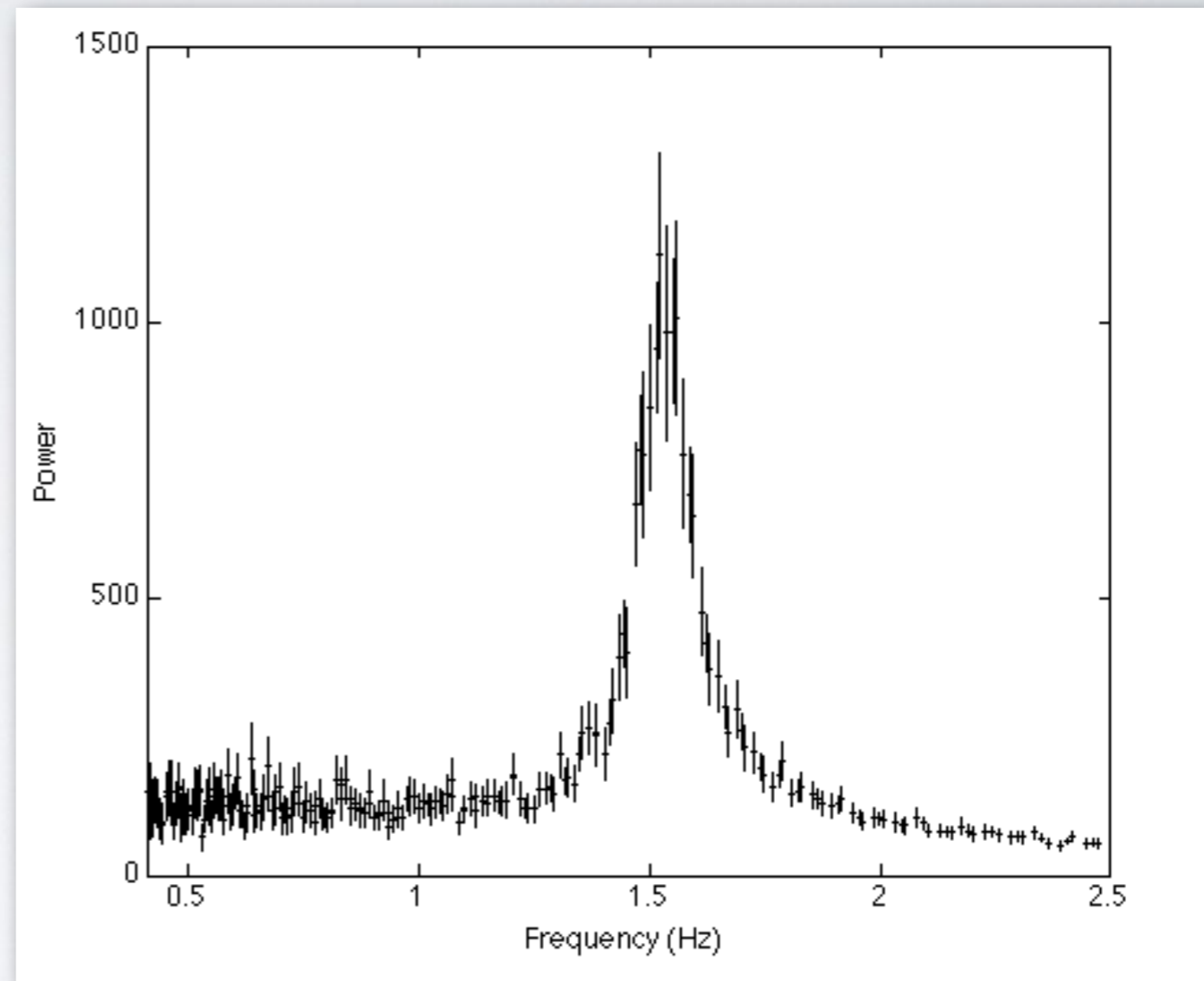
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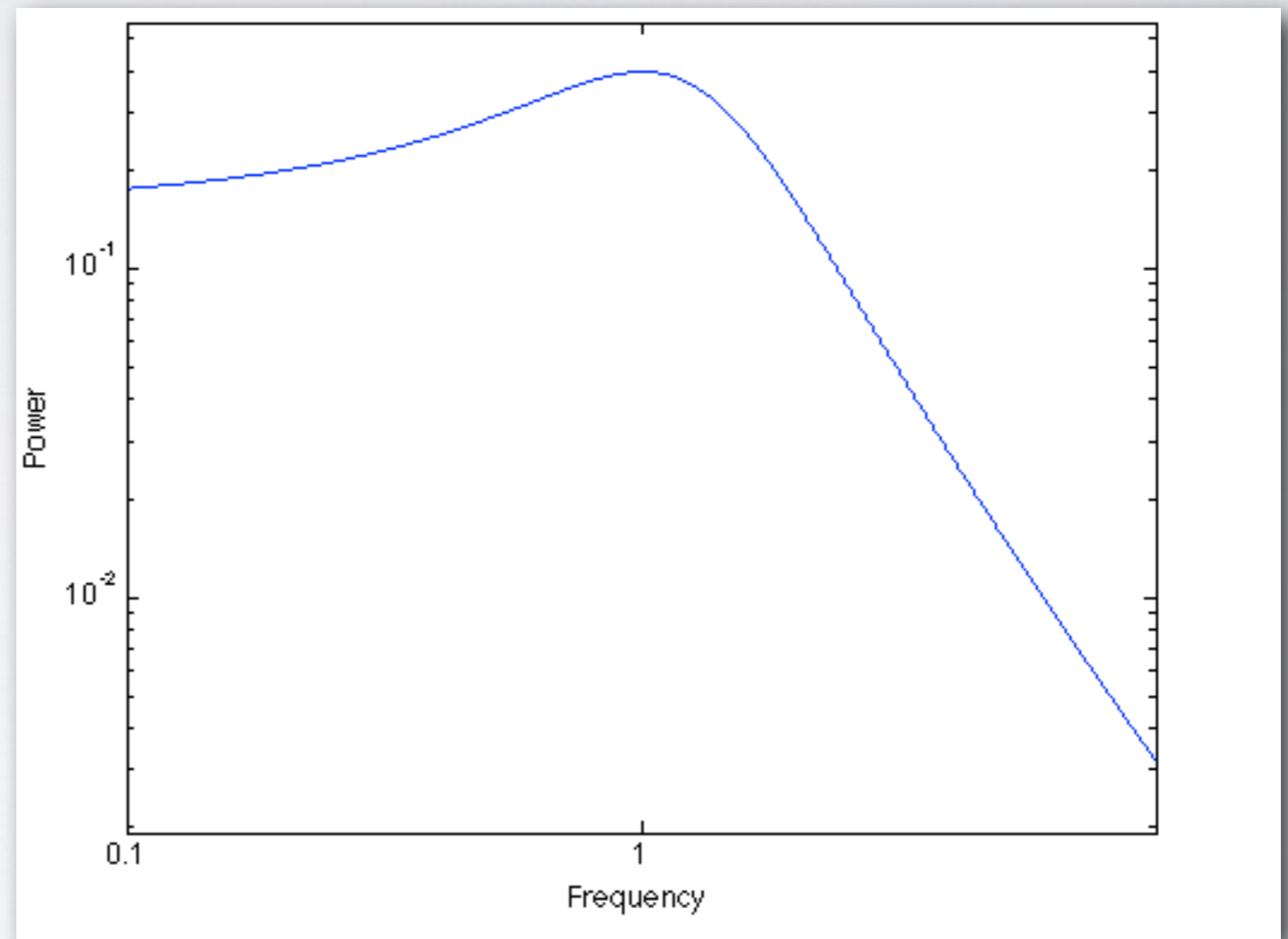
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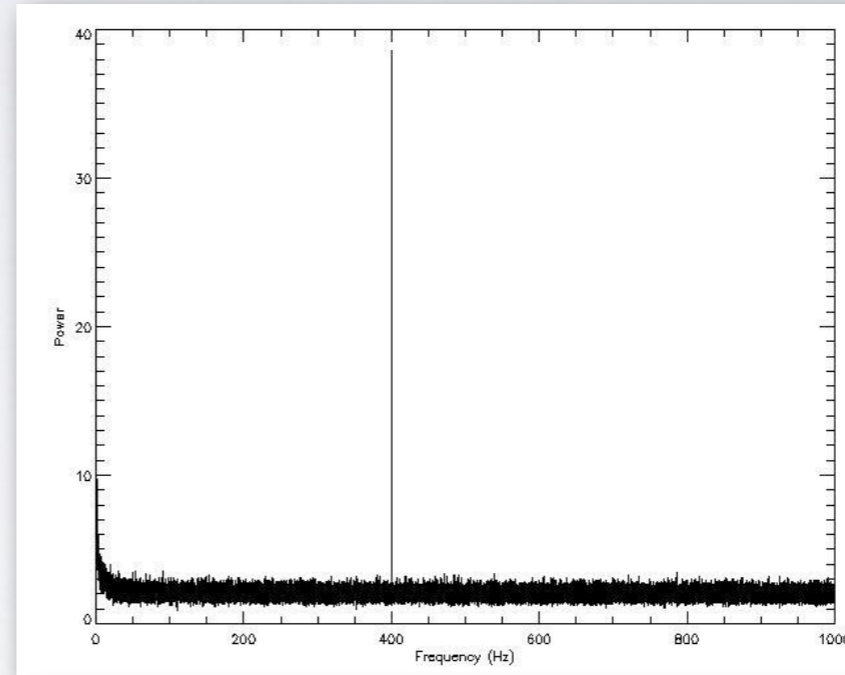
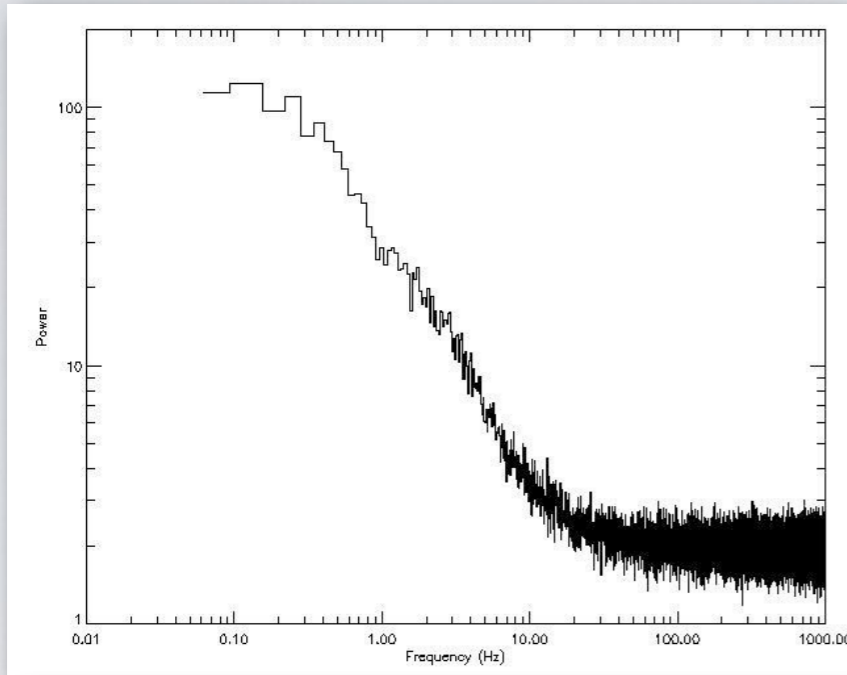


MAIN TYPES OF SIGNALS

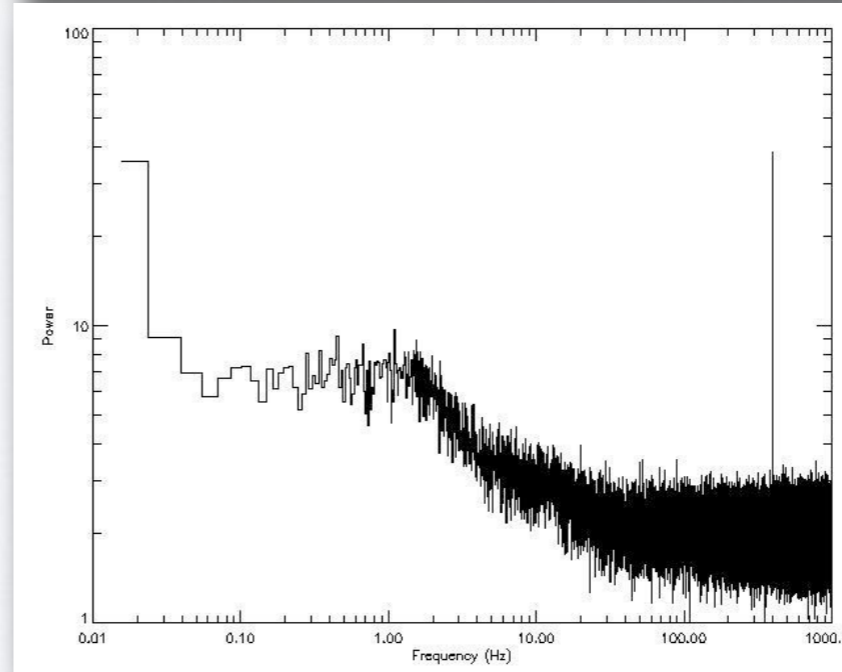
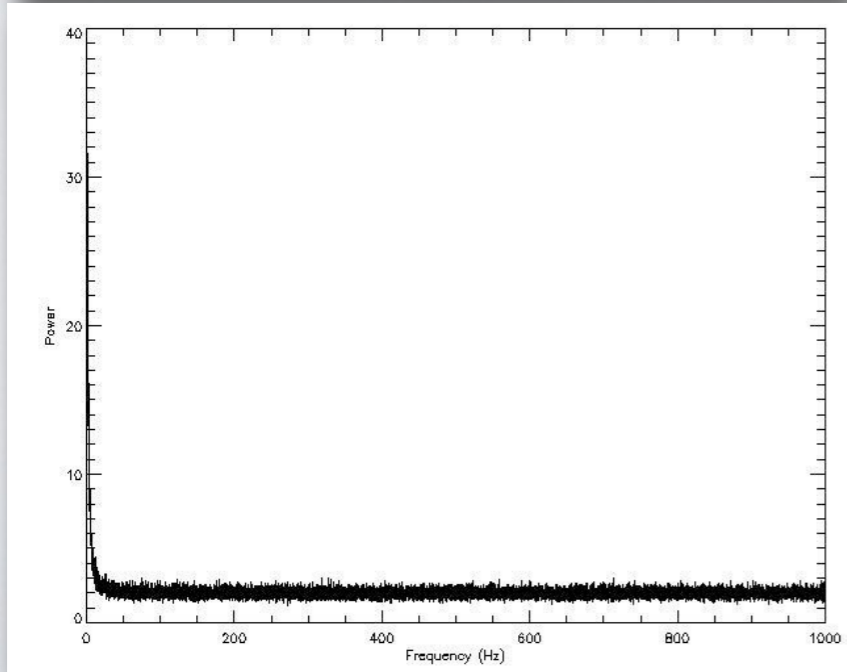
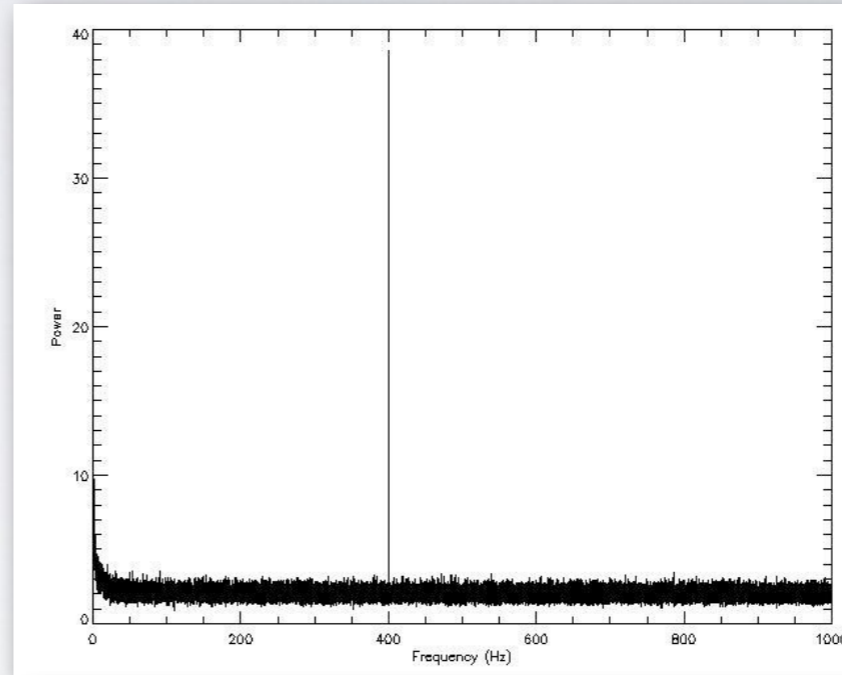
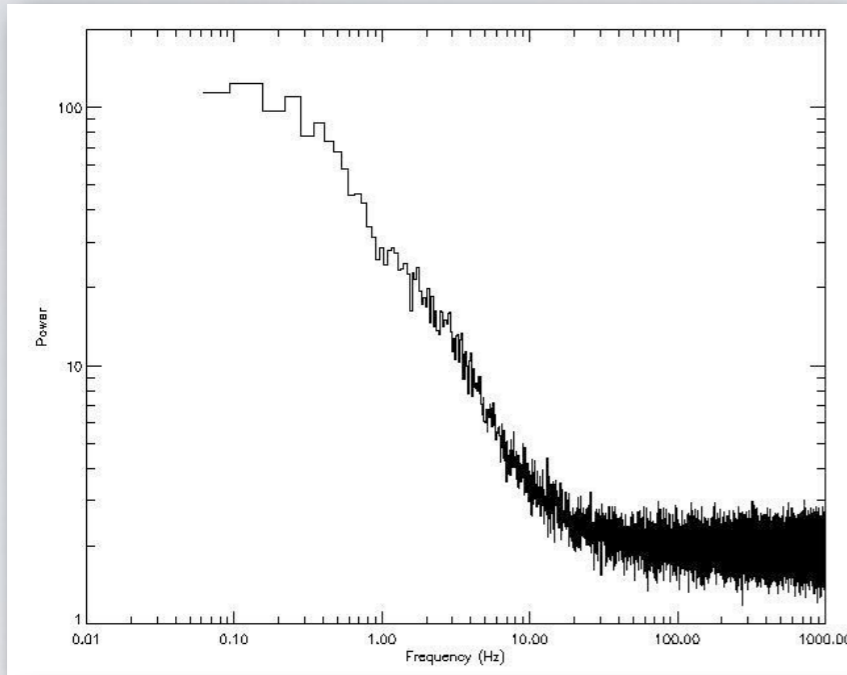
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A WORD ON REPRESENTATION



A WORD ON REPRESENTATION

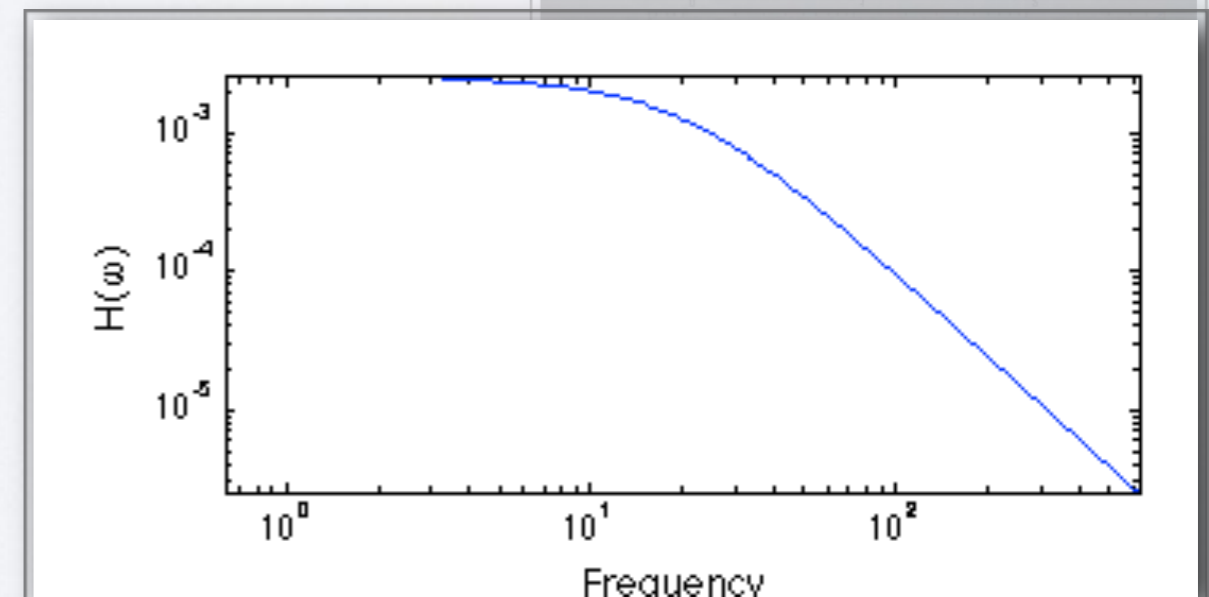
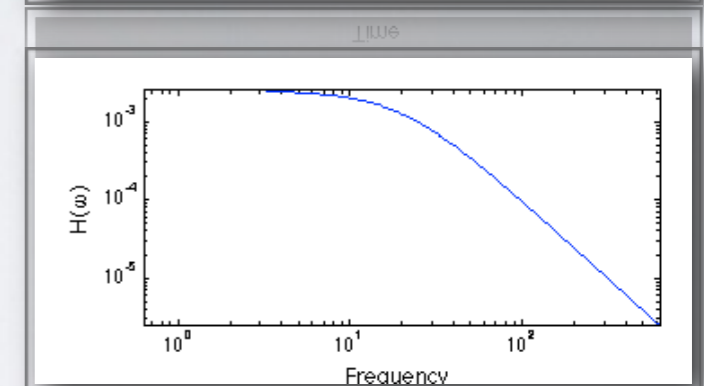
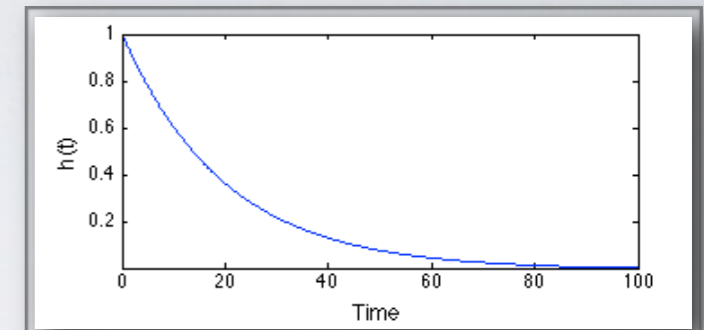


THE LORENTZIAN (ZERO-CENTERED)

- ❖ Power spectrum of a one-sided exponential

$$L(\nu; N, \Delta) = \frac{\Delta}{2\pi} \frac{1}{\nu^2 + \left(\frac{\Delta}{2}\right)^2}$$

- ❖ Good for modeling broadband noise components (flat-top)

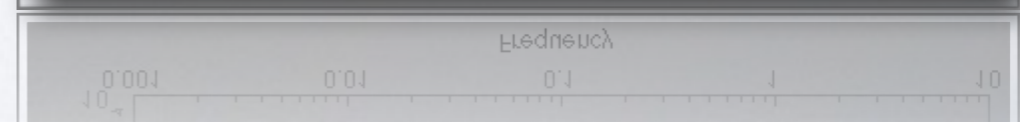
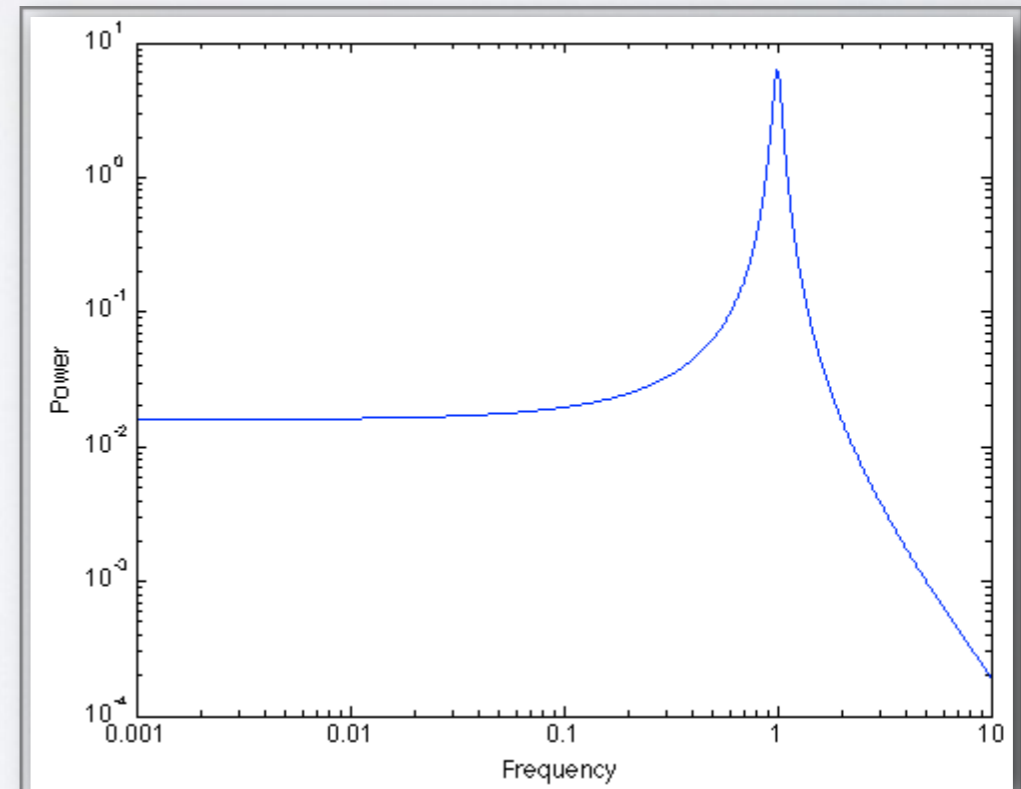


THE LORENTZIAN

- ❖ Centroid of Lorentzian not at zero

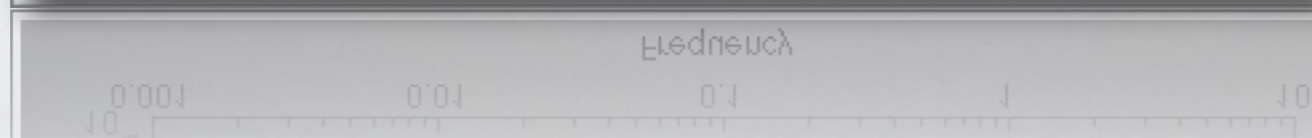
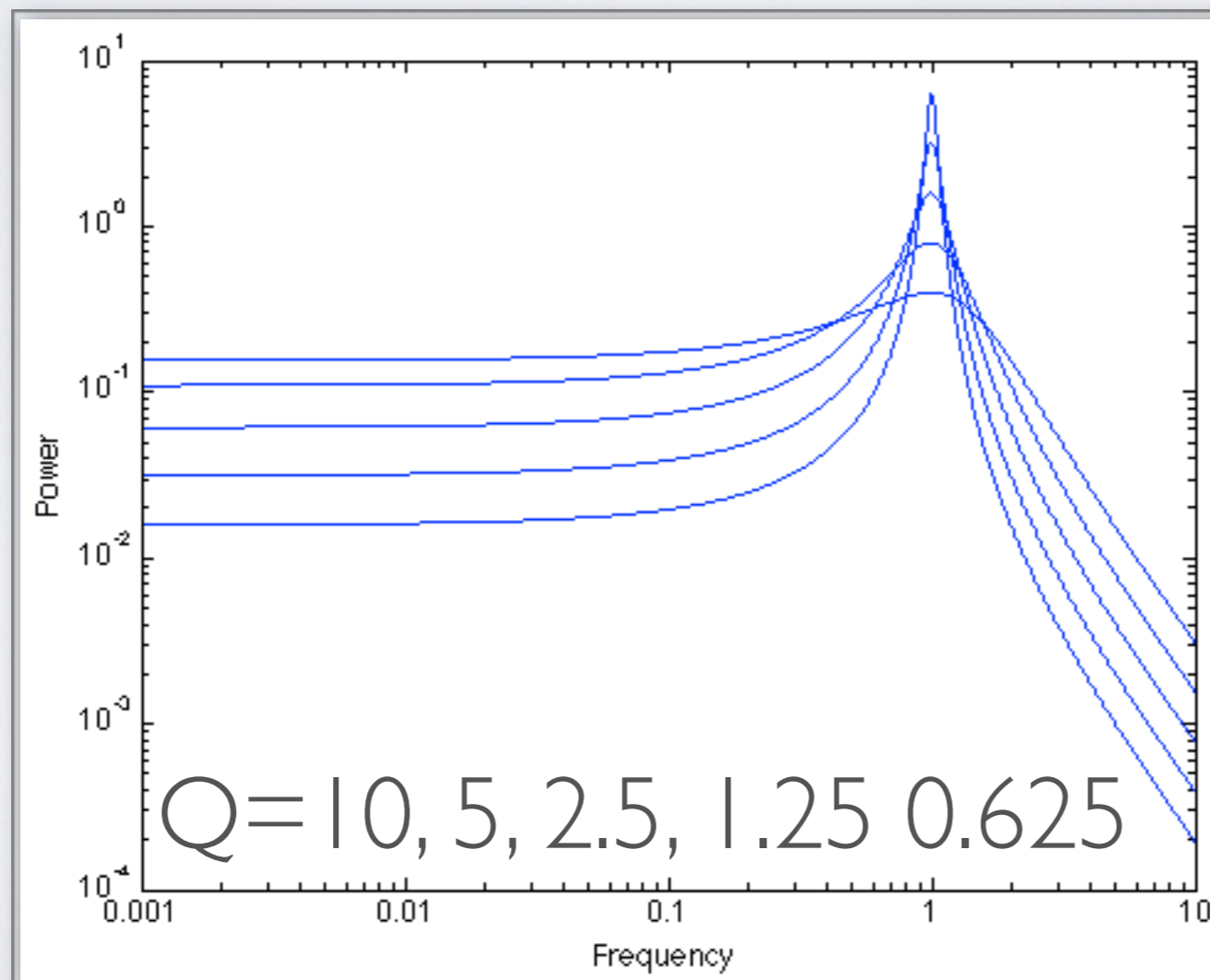
$$L(\nu; N, \nu_0, \Delta) = \frac{\Delta}{2\pi} \frac{1}{(\nu - \nu_0)^2 + \left(\frac{\Delta}{2}\right)^2}$$

- ❖ Good for modeling Quasi-Periodic Peaks



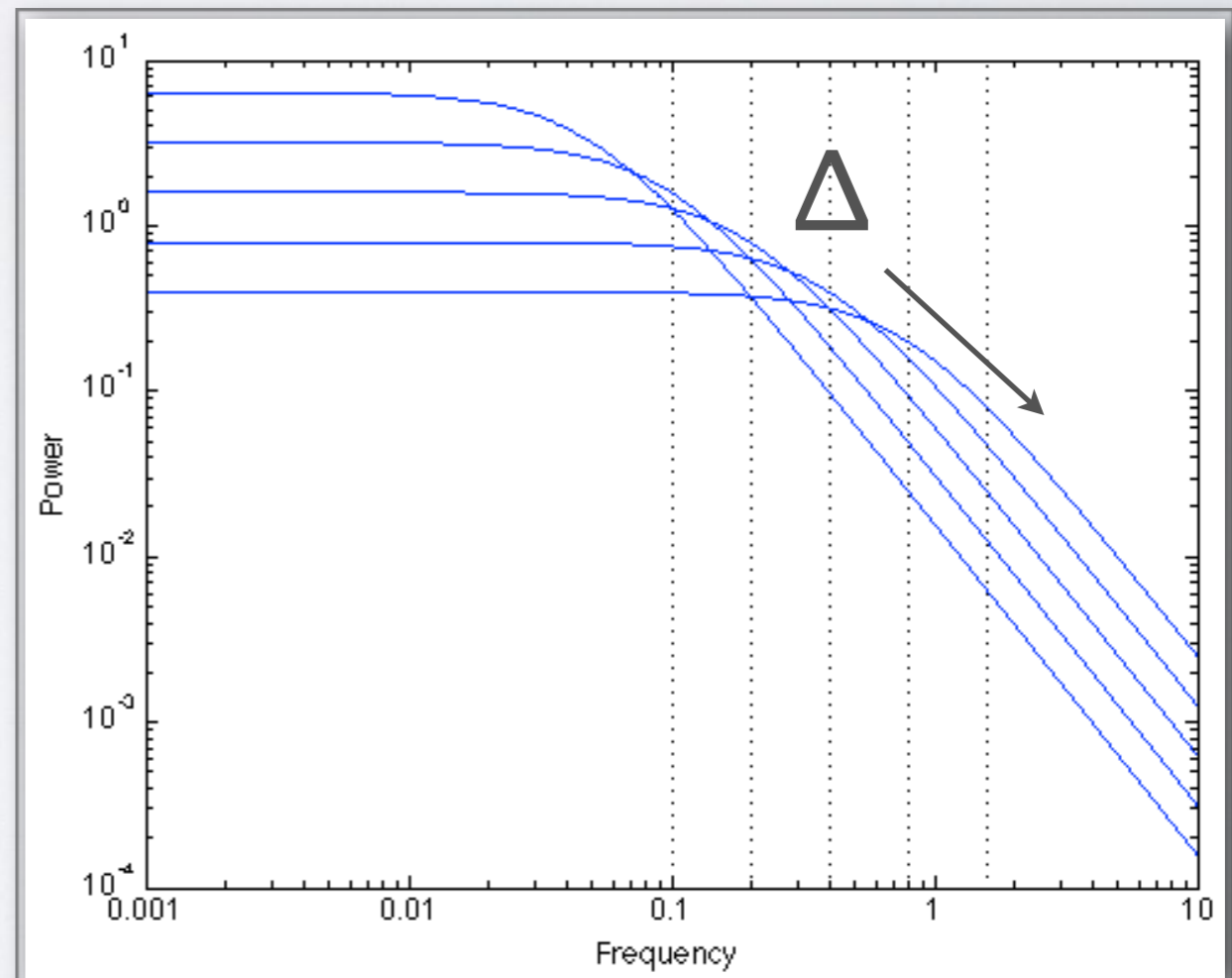
THE QUALITY FACTOR Q

- To quantify the coherence of a component $Q = \frac{\nu_0}{\Delta}$



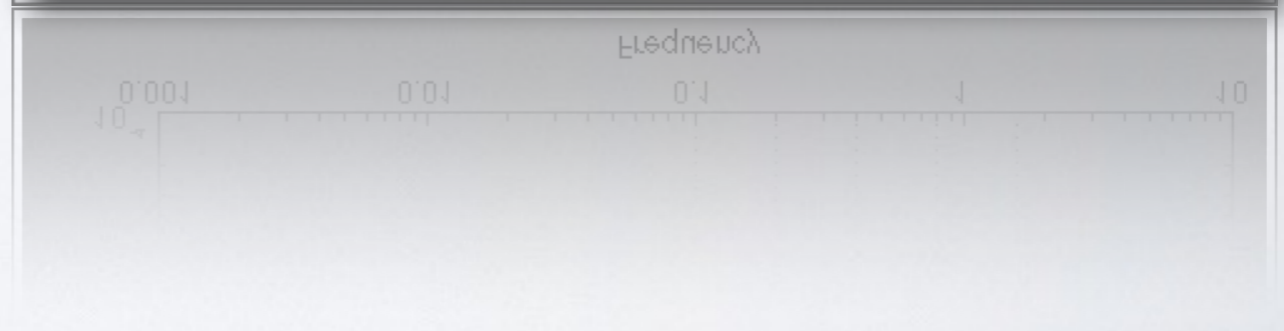
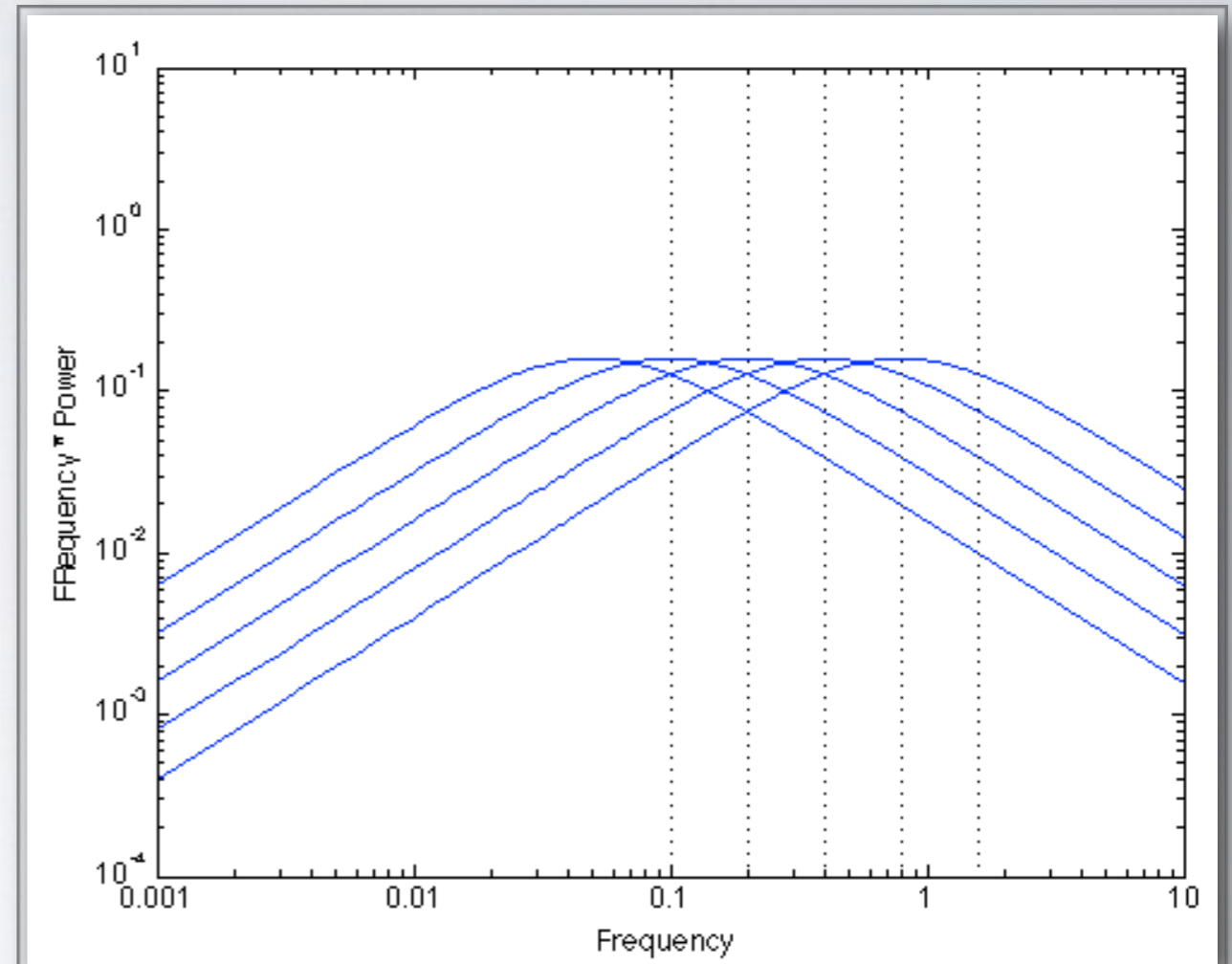
Q=0: THE PEAK WITHOUT QUALITY

- ❖ Here $\nu_0 = 0$, equal N
- ❖ Notice position of the break
- ❖ Factor of two higher



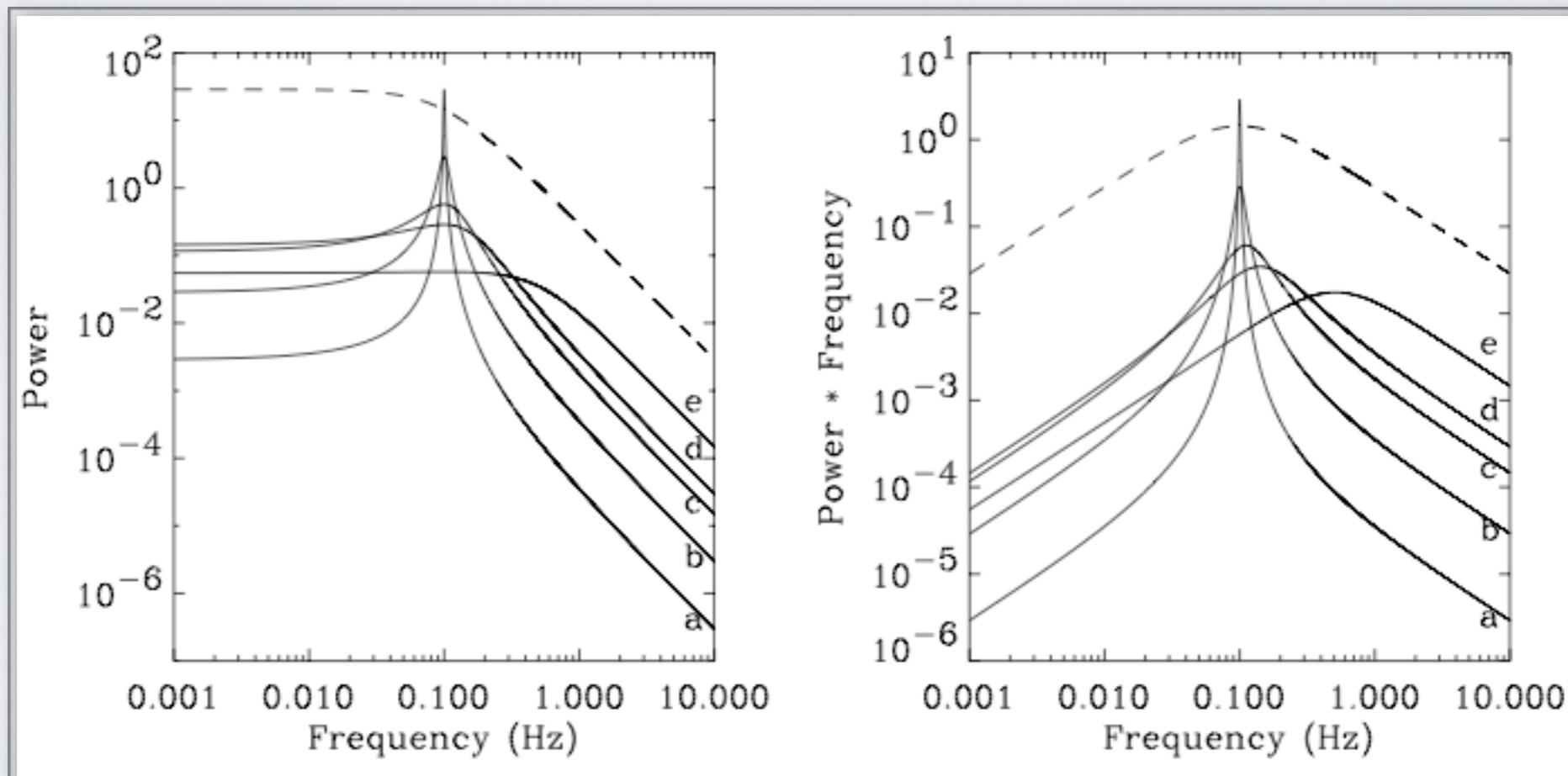
BETTER REPRESENTATION

- ❖ In νP_ν the effect is the same
- ❖ Better value is $\Delta/2$
- ❖ But... how do I treat things homogeneously and how do I treat peaked noise?



CHARACTERISTIC FREQUENCY

- We can use the peak in νP_ν $\nu_{max} = \sqrt{\nu_0^2 + \frac{\Delta^2}{2}}$

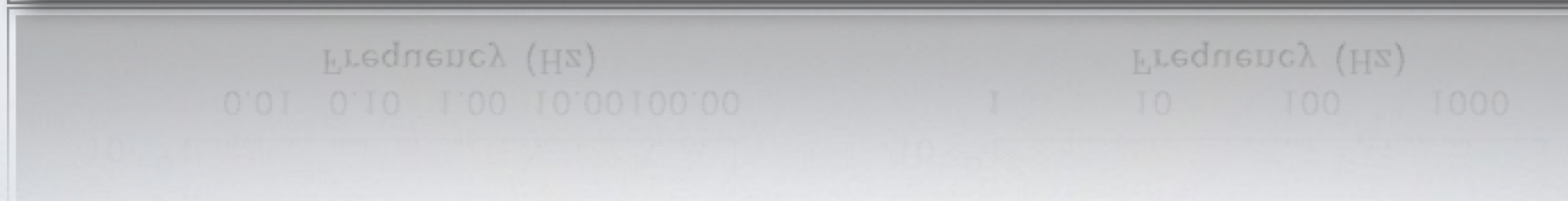
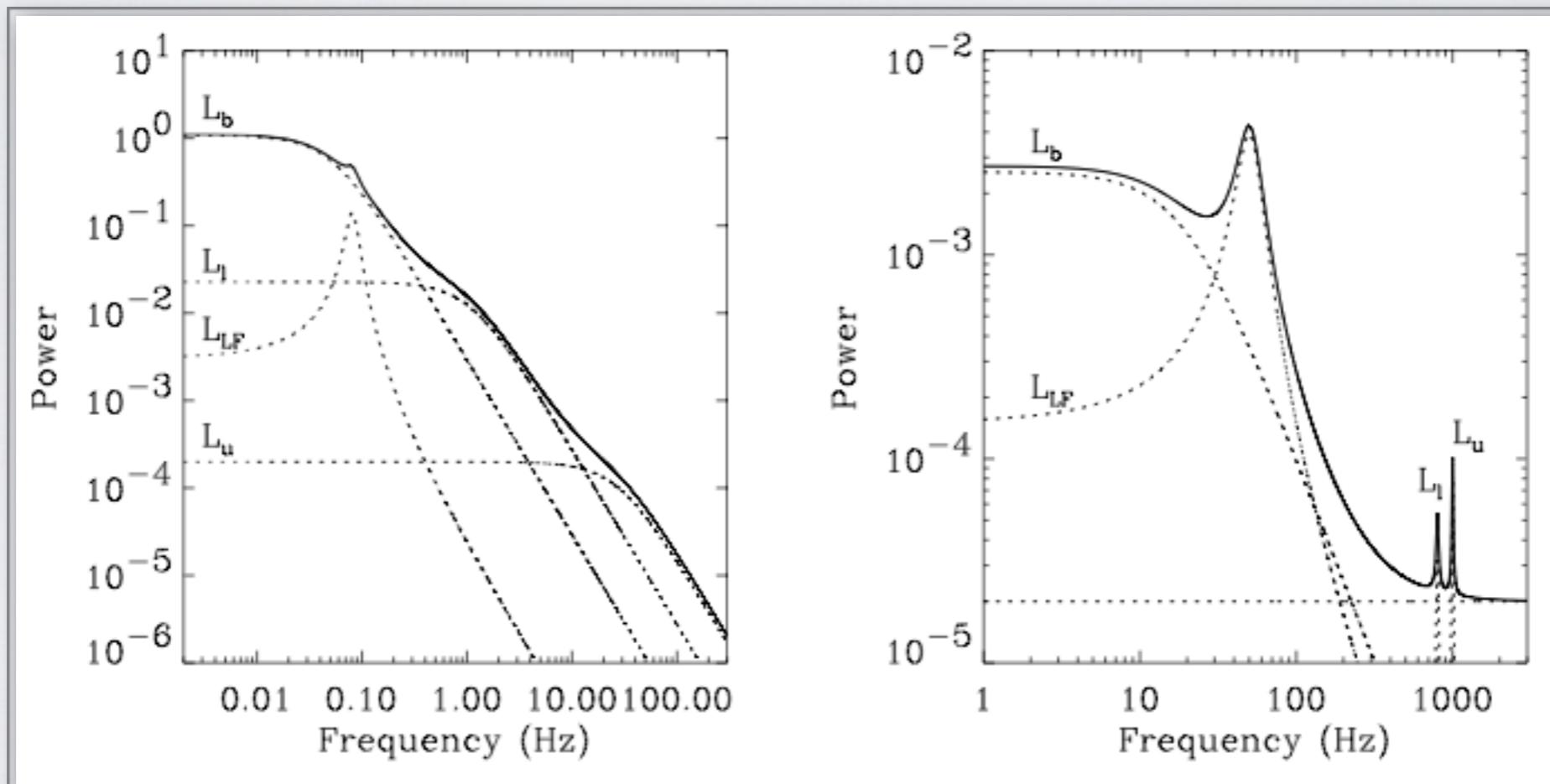


Ελεγχος (Hz)
0.001 0.010 0.100 1.000 10.000

Ελεγχος (Hz)
0.001 0.010 0.100 1.000 10.000

LORENTZIAN DECOMPOSITION

- ✦ With these tools we can fit power spectra



NO PHYSICAL BACKING (YET)

- ❖ Power spectrum of a damped oscillator
- ❖ Also called Cauchy distribution
- ❖ Even if it looks like a Lorentzian, it might not be a Lorentzian



DEALING WITH GAPS

- ❖ Some solutions are obvious:
 - ❖ Welch method (skip gaps)
 - ❖ Zero padding (or local average)
- ❖ Other methods are available: Lomb-Scargle
 - ❖ Good for general uneven sampling
 - ❖ Equivalent to linear least-square fit to $\sin + \cos$
 - ❖ Statistically robust

LOMB-SCARGLE PERIODOGRAM

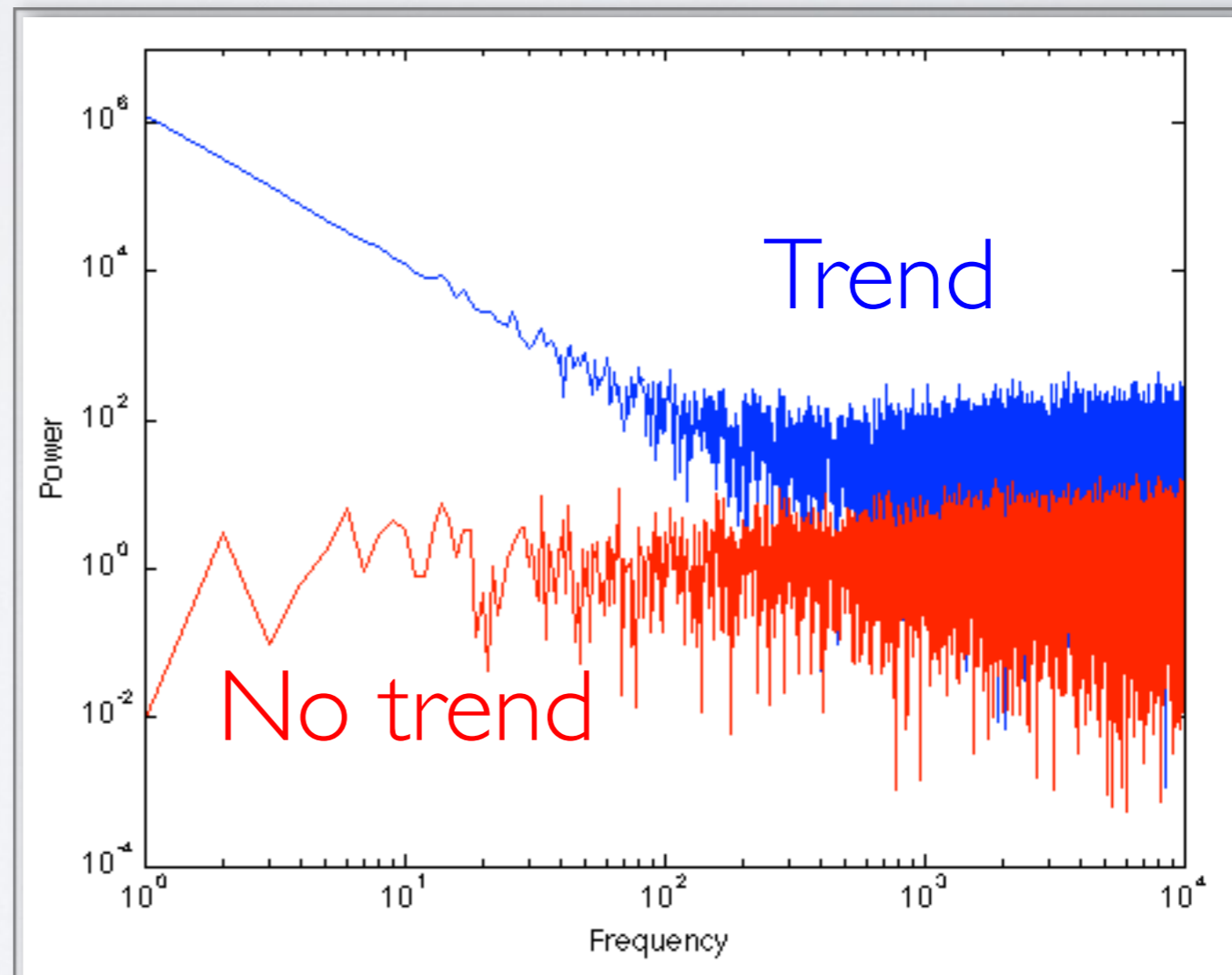
- ❖ h_j sampled at t_j

$$P_N(\omega) = \frac{1}{2\sigma^2} \left\{ \frac{[\sum_j (h_j - \bar{h}) \cos \omega(t_j - \tau)]^2}{\sum_j \cos^2 \omega(t_j - \tau)} + \frac{[\sum_j (h_j - \bar{h}) \sin \omega(t_j - \tau)]^2}{\sum_j \sin^2 \omega(t_j - \tau)} \right\}$$

- ❖ where: $\tan(2\omega\tau) = \frac{\sum_j \sin 2\omega t_j}{\sum_j \cos 2\omega t_j}$ ensures shift independence
- ❖ Powerful method: it can go beyond “Nyquist”

BEWARE OF TRENDS!

- ❖ A trend is a modification to the window
- ❖ Must be de-trended
- ❖ Same about possible drop outs

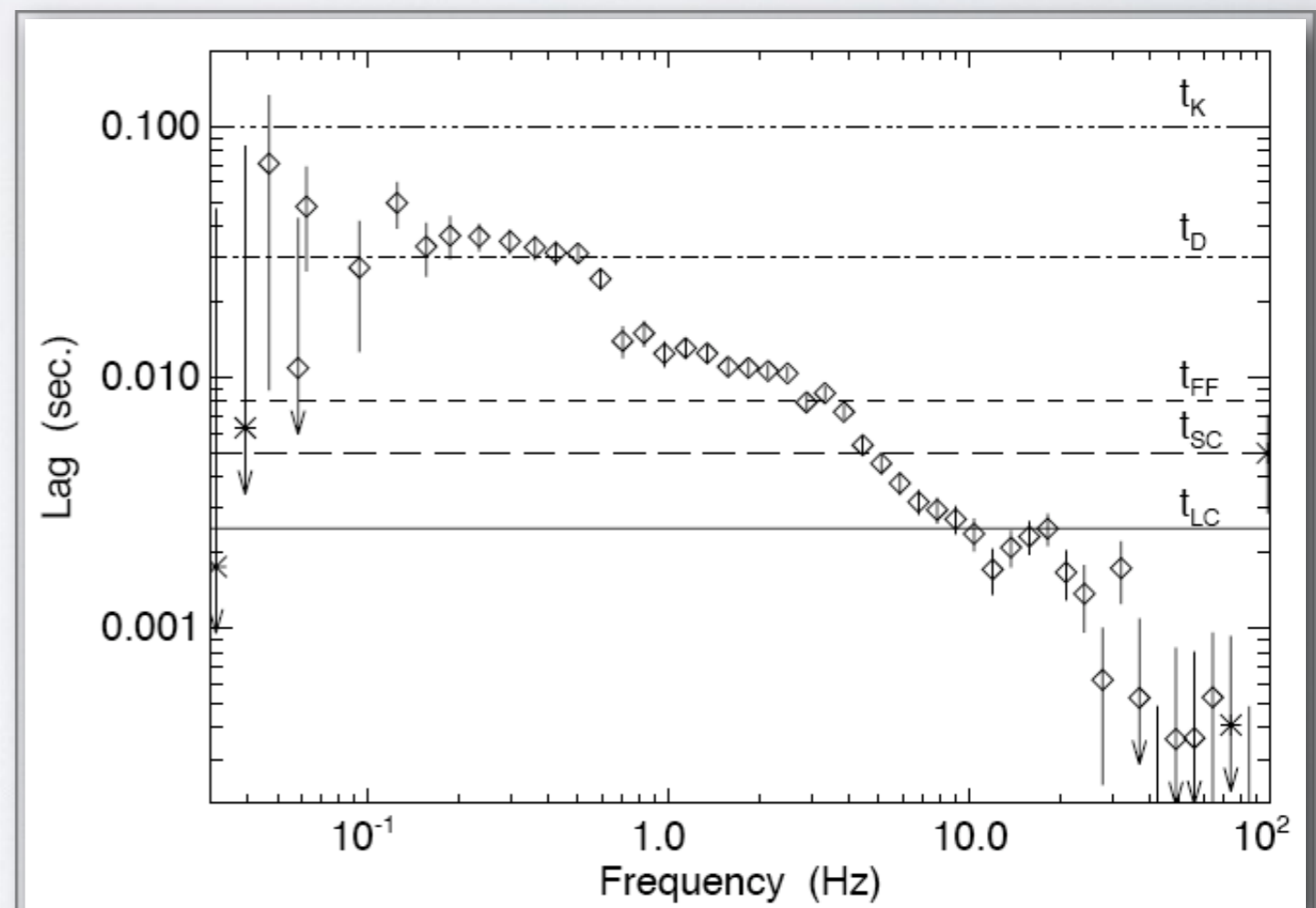


CROSS-SPECTRUM

- ❖ Power spectrum: amplitudes of the FFT
- ❖ We throw away the phases
- ❖ If we take *two* time series $f(t)$ & $g(t)$, the phases make more sense
- ❖ Cross-spectrum: $C_{f,g}(\nu) = F_f^*(\nu) \times F_g(\nu)$
- ❖ If $f=g$, it becomes the power spectrum
- ❖ What is it useful for?

PHASE/TIME LAGS

- ❖ The phases give us the phase delay between the two time series
- ❖ Not easy to interpret, can be linked to physical models
- ❖ Time lags: phase φ/ν
- ❖ Additional technical details (not shown)



AUTO/CROSS-CORRELATION

- ❖ The power spectrum is the FT of the autocorrelation

$$\text{Corr}(g, g) = \int_{-\infty}^{\infty} g(t + \tau)g(\tau)d\tau \iff |G(f)|^2$$

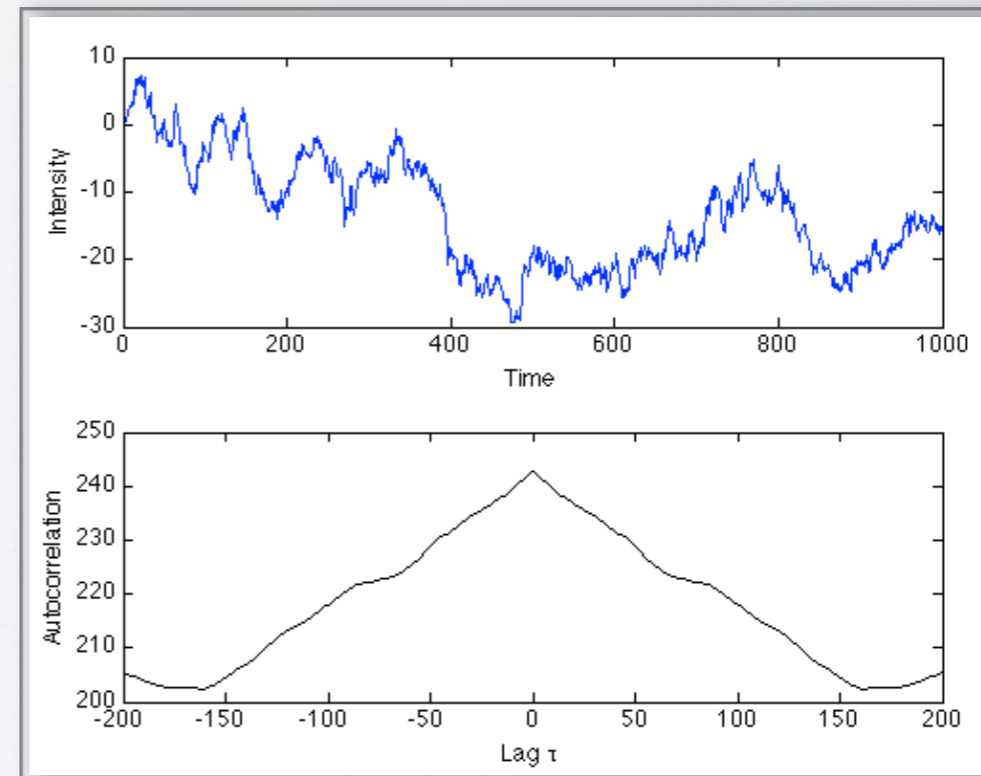
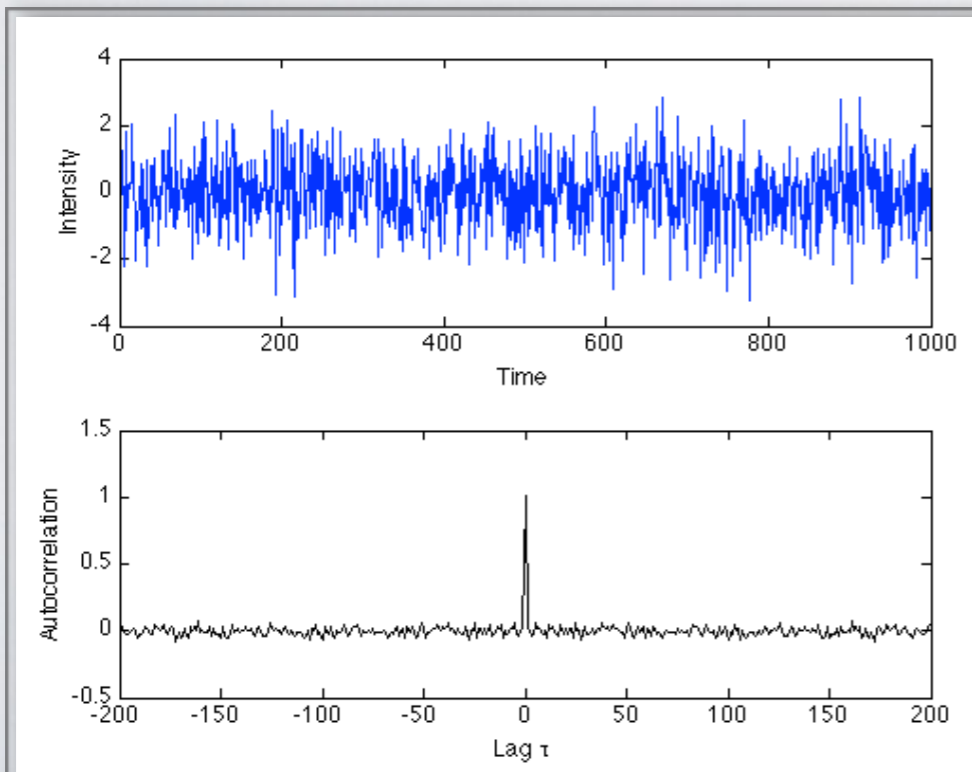
- ❖ Autocorrelation is real and even, power spectrum is real and even
- ❖ The cross spectrum is the FT of the crosscorrelation
- ❖ Power- an cross-spectrum contain more information (if you can afford them because of statistics)

AUTOCORRELATION

- ❖ Uncorrelated noise: ACF is zero everywhere but at $\tau=0$ [variance]

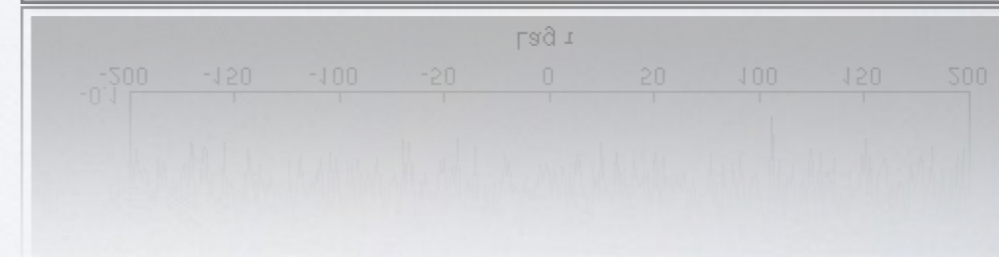
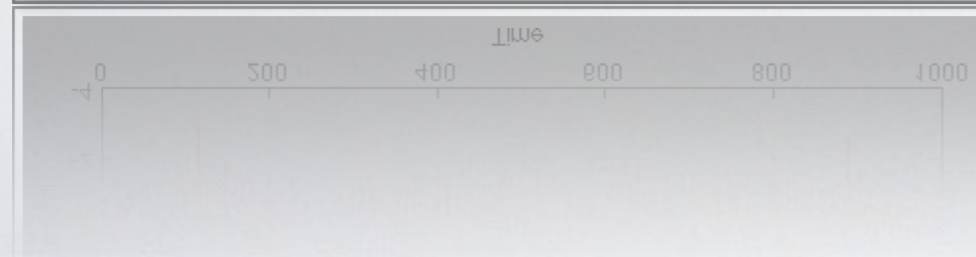
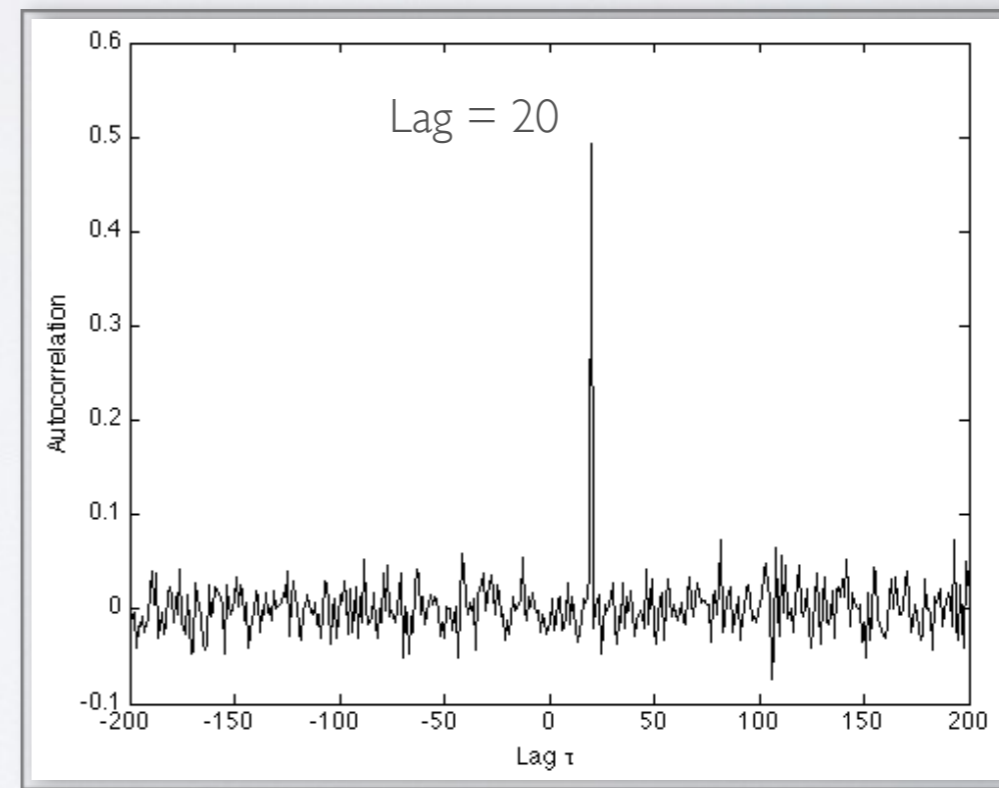
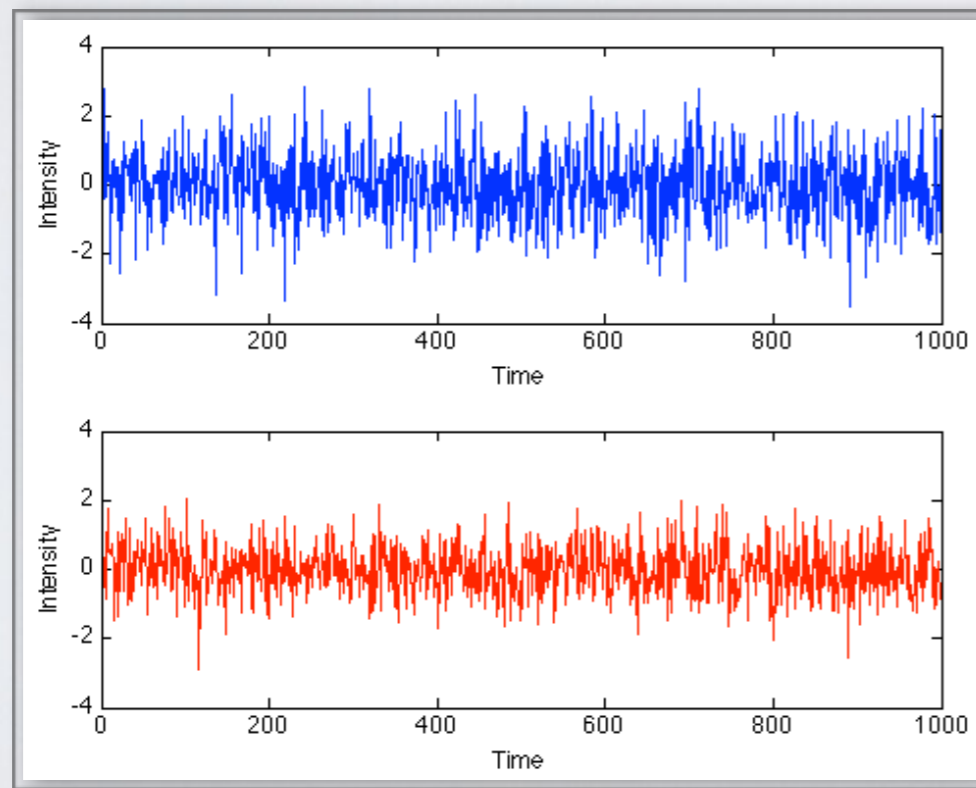
$$\text{Corr}(g, g) = \int_{-\infty}^{\infty} g(t + \tau)g(\tau)d\tau$$

- ❖ Biased ACF: dividing by N
- ❖ Unbiased ACF: dividing by $N-|m|$



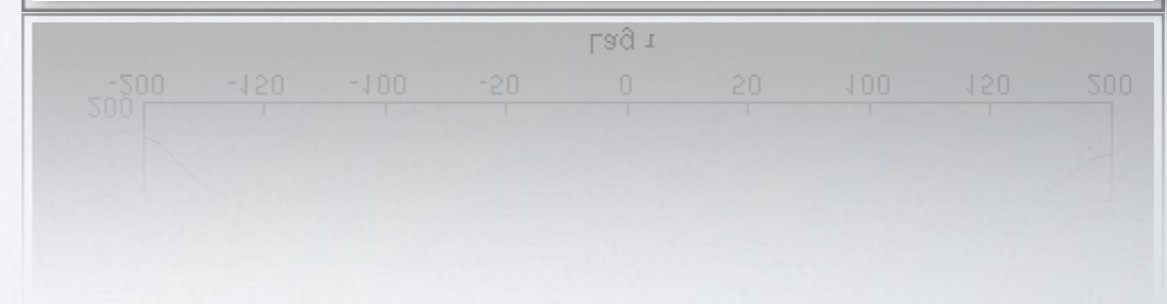
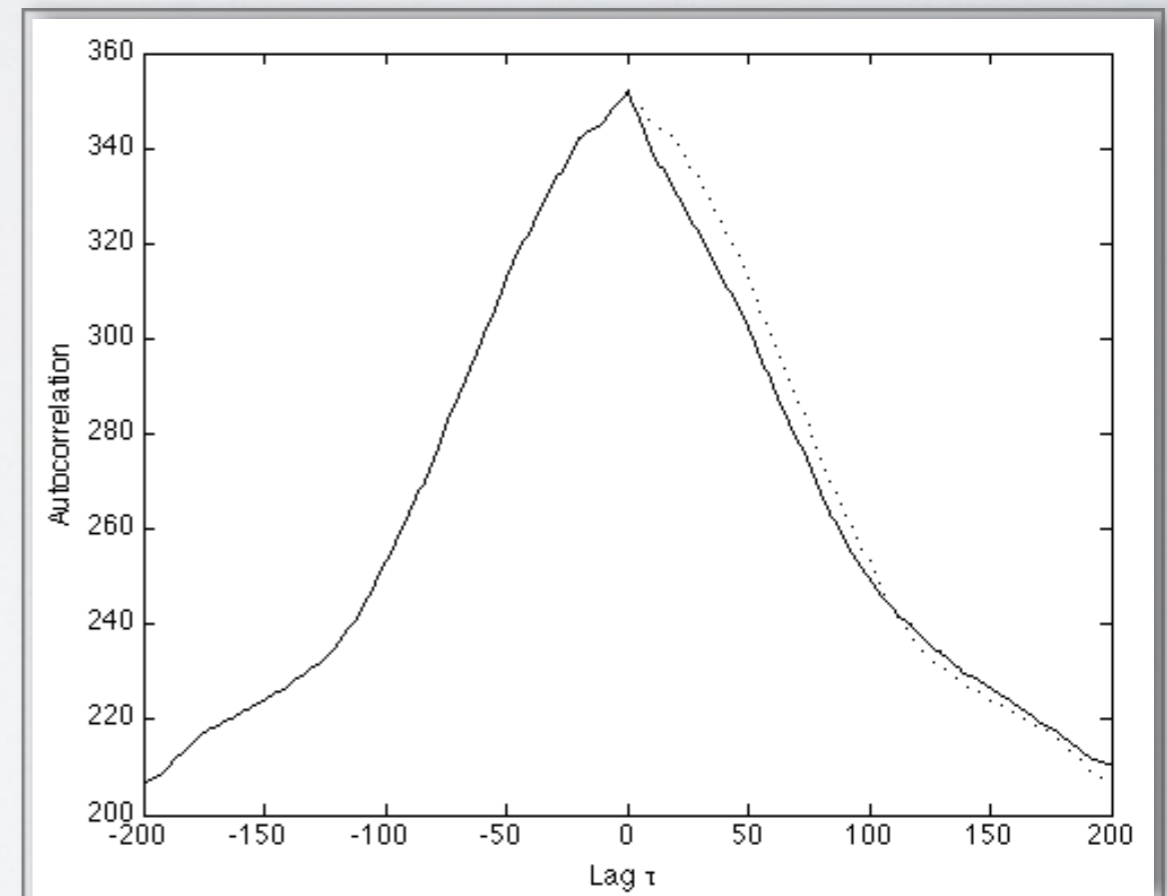
CROSSCORRELATION

- ❖ Uncorrelated series: CCF is zero everywhere
- ❖ Simple shift: peak somewhere



WHEN DO I HAVE A LAG?

- ❖ *“The CCF peaks at 0, therefore there is no measurable lag”*
- ❖ NO!
- ❖ CCF is a superposition of sinusoids of different periods
- ❖ Any asymmetry implies a lag



COHERENT SIGNALS: BARYCENTRIC CORR.

- ❖ The Earth moves and rotates, the satellite also moves
- ❖ This has an effect on the period (doppler modulation)..
- ❖ .. and on the absolute phase
- ❖ Times are corrected to the barycenter of the solar system
- ❖ Standard routines and ephemeris
- ❖ Not relevant for aperiodic signals

PERIOD FOLDING I: χ^2 TEST

- ❖ Photon arrival times t_j
- ❖ For trial period produce phases $\phi_j = \text{Frac} \left(\frac{t_j}{P} \right)$
- ❖ Put photon in appropriate phase bin
- ❖ Test vs. constancy (χ^2)
- ❖ If time bins and not times, easy to generalize
- ❖ Problem: binning and statistics (few photons?)

PERIOD FOLDING II: Z^2 TEST

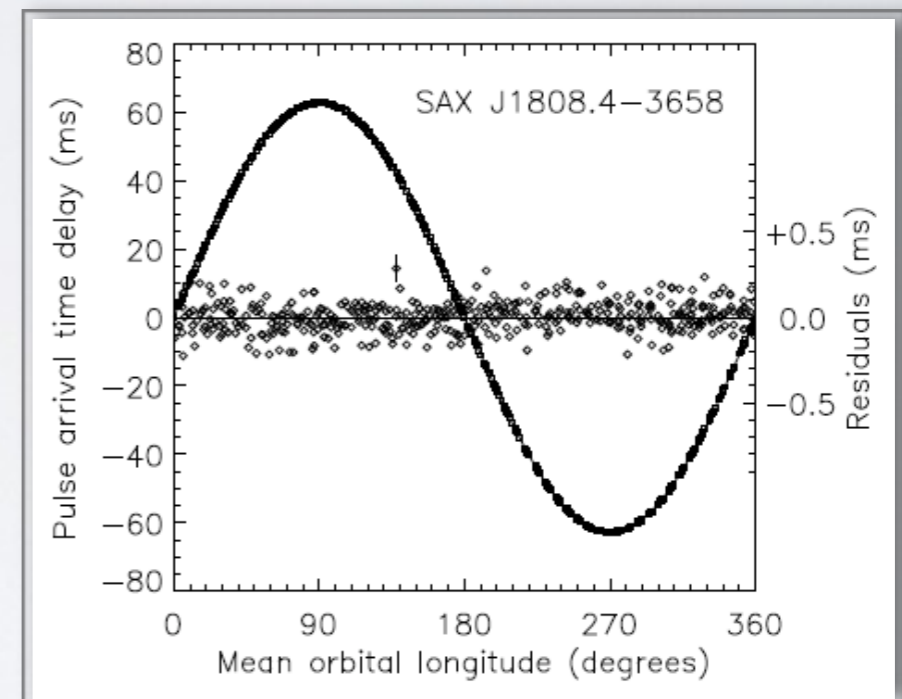
- ❖ Photon arrival times t_j
- ❖ For trial period produce phases $\phi_j = \text{Frac} \left(\frac{t_j}{P} \right)$
- ❖ Compute
$$Z_n^2 = \frac{2}{N} \sum_{k=1}^n \left[\left(\sum_{j=1}^N \cos k\phi_j \right)^2 + \left(\sum_{j=1}^N \sin k\phi_j \right)^2 \right]$$

where n is the desired number of harmonics

- ❖ Z is distributed as a χ^2 with $2n$ d.o.f.
- ❖ Good for small number of photons [Rayleigh test]

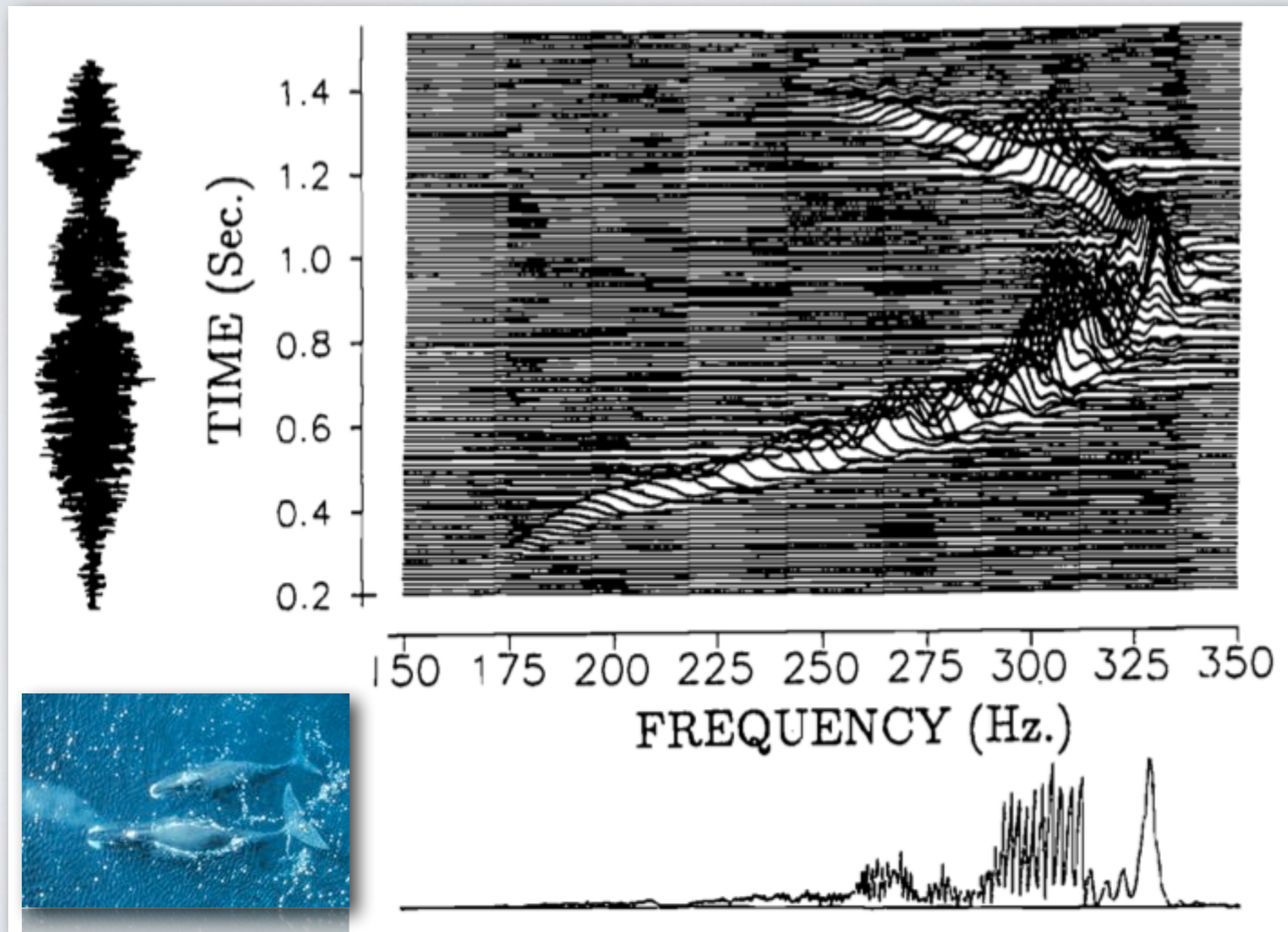
COMPLICATIONS

- ❖ There can be a significant period derivative
- ❖ If your pulsar is in a binary system, there is Doppler effect
- ❖ Easy to lose a pulsation
- ❖ Power spectrum smeared, folding as well
- ❖ Must factorize possible orbit in the solution
- ❖ Many free parameters



MORE ON TIME-FREQUENCY

Bowhead whale



ALTERNATIVE TECHNIQUES

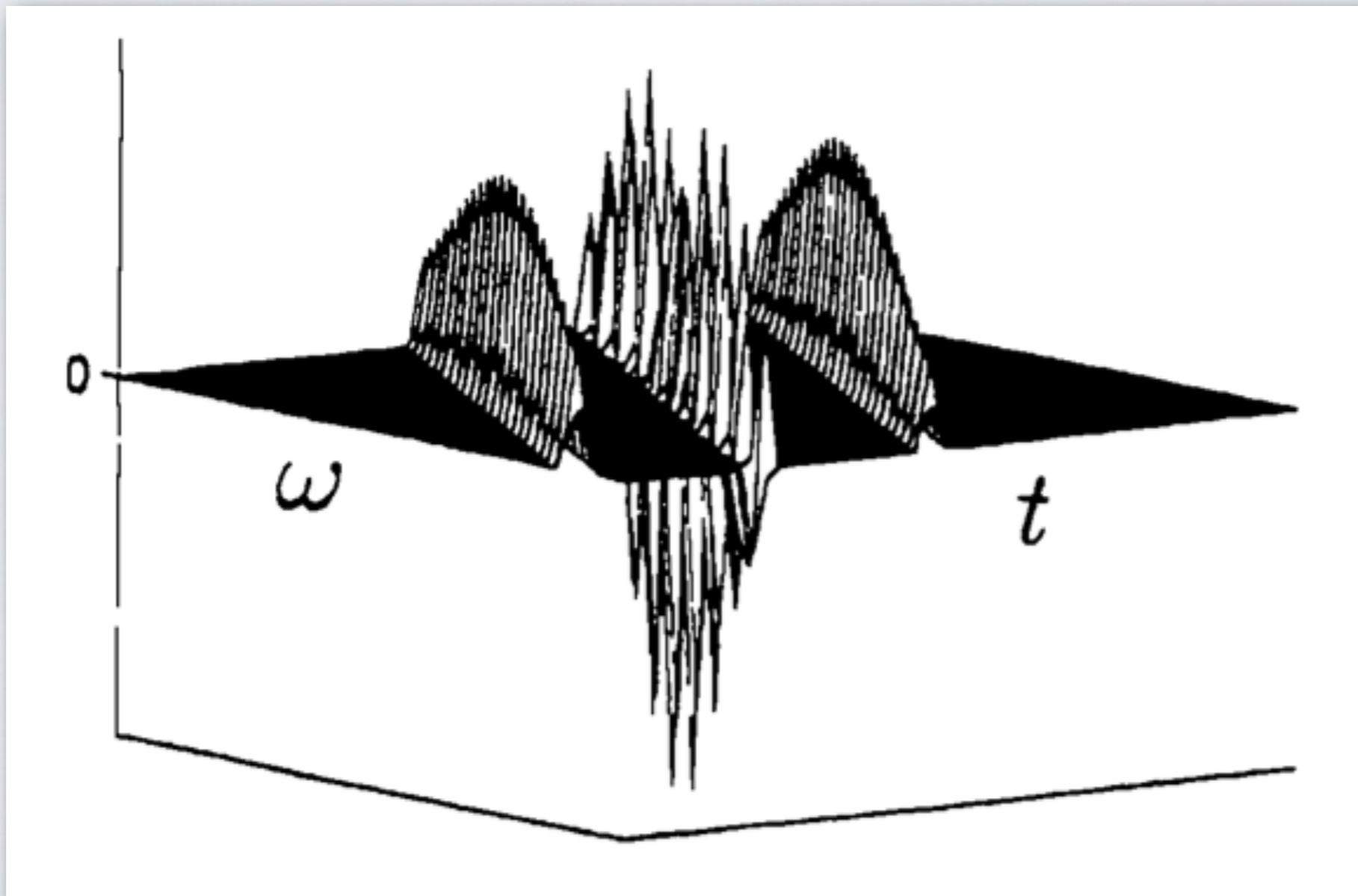
- The Wigner Distribution

$$W(t, \omega) = \frac{1}{2\pi} \int s^*(t - \frac{1}{2}\tau) s(t + \frac{1}{2}\tau) e^{-i\tau\omega} d\tau$$

- Signal in the past by the signal in the future!
- Problem: only for Gaussian chirps W is everywhere positive
- You can beat the uncertainty principle, but at the cost..
- ... of generating additional monstruosities

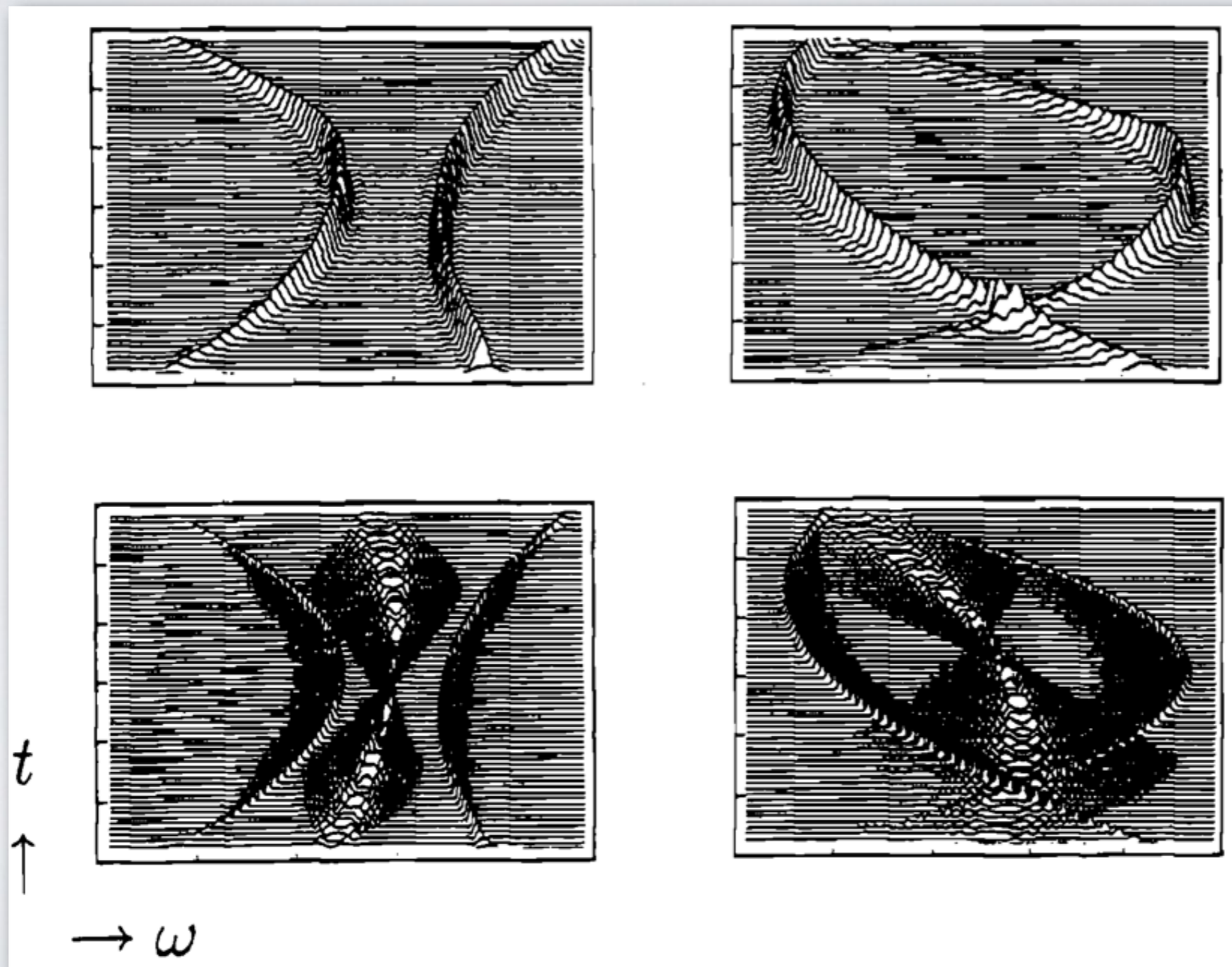
THE WIGNER DISTRIBUTION

Signal: sum of two chirps



THE WIGNER DISTRIBUTION

Comparison with the spectrogram



COHEN'S KERNEL

ALL time-frequency representations come from:

$$C(t, \omega) = \frac{1}{4\pi^2} \int \int \int s^*(u - \frac{1}{2}\tau) s(u + \frac{1}{2}\tau) \phi(\theta, \tau) e^{-i\theta t - i\tau\omega + i\theta u} du d\tau d\theta$$

Where $\phi(\theta, \tau)$ is the kernel

Changing the kernel you change the representation

The properties of the representation depend on the properties of the kernel

Name	Kernel: $\phi(\theta, \tau)$	Distribution: $C(t, \omega)$
General class (Cohen ^[125])	$\phi(\theta, \tau)$	$\frac{1}{4\pi^2} \iiint e^{-j\theta t - j\tau\omega + j\theta u} \phi(\theta, \tau) s^*(u - \frac{1}{2}\tau) s(u + \frac{1}{2}\tau) du d\tau d\theta$
Wigner ^[584]	1	$\frac{1}{2\pi} \int e^{-j\tau\omega} s^*(t - \frac{1}{2}\tau) s(t + \frac{1}{2}\tau) d\tau$
Margenau-Hill ^[358]	$\cos \frac{1}{2}\theta\tau$	$\text{Re} \frac{1}{\sqrt{2\pi}} s(t) S^*(\omega) e^{-jt\omega}$
Kirkwood ^[305] Rihaczek ^[484]	$e^{j\theta\tau/2}$	$\frac{1}{\sqrt{2\pi}} s(t) S^*(\omega) e^{-jt\omega}$
Born-Jordan ¹ (Cohen ^[125])	$\frac{\sin \frac{1}{2}\theta\tau}{\frac{1}{2}\theta\tau}$	$\frac{1}{2\pi} \int \frac{1}{ \tau } e^{-j\tau\omega} \int_{t- \tau /2}^{t+ \tau /2} s^*(u - \frac{1}{2}\tau) s(u + \frac{1}{2}\tau) du d\tau$
Page ^[419]	$e^{j\theta \tau }$	$\frac{\partial}{\partial t} \left \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t s(t') e^{-j\omega t'} dt' \right ^2$
Choi-Williams ^[117]	$e^{-\theta^2\tau^2/\sigma}$	$\frac{1}{4\pi^{3/2}} \iint \frac{1}{\sqrt{\tau^2/\sigma}} e^{-\sigma(u-t)^2/\tau^2 - j\tau\omega} s^*(u - \frac{1}{2}\tau) s(u + \frac{1}{2}\tau) du d\tau$
Spectrogram	$\int h^*(u - \frac{1}{2}\tau) e^{-j\theta u} h(u + \frac{1}{2}\tau) du$	$\left \frac{1}{\sqrt{2\pi}} \int e^{-j\omega\tau} s(\tau) h(\tau - t) d\tau \right ^2$
Zhao-Atlas-Marks ^[626]	$g(\tau) \tau \frac{\sin a\theta\tau}{a\theta\tau}$	$\frac{1}{4\pi a} \int g(\tau) e^{-j\tau\omega} \int_{t- \tau a}^{t+ \tau a} s^*(u - \frac{1}{2}\tau) s(u + \frac{1}{2}\tau) du d\tau$
Positive ² (Cohen, Posch, Zaparovanny ^[127, 128])	see Chapter 14	$ S(\omega) ^2 s(t) ^2 \Omega(u, v)$

A WORD ON WAVELETS

- The resolution element in time-frequency cannot be made smaller than the minimum
- However, there is no reason why it should be of the same shape
- You can adapt it to what you need where you need it

