Solutions to Quiz 1

1. For a positive integer n, with the usual rules for addition and multiplication in $\mathbb{Z}/n$, show that multiplication distributes over addition.

Solution: Given integers a and b we perform division by n

$$a = cn + d \text{ and } b = en + f$$

with d and f non-negative integers less than n.

If $(e + f) = gn + h$ is the division of $e + f$ by n, then

$$a + b = (c + d)n + e + f = (c + d + g)n + h$$

is the division of $a + b$ by n. Hence, whether we take remainder modulo n before addition or after addition, the result is the same.

Similarly, if $ef = kn + m$ is the division of ef by n, then

$$ab = (cen + cf + ed)n + ef = (cen + cf + ed + k)n + m$$

is the division of ab by n. Hence, whether we take the remainder modulo n before multiplication of after multiplication, the result is the same.

In other words, taking remainder modulo n before or after an arithmetic opration has the same result.

Now multiplication distributes over addition in integers. So if we look at a, b and c in the subset $\{0, 1, \ldots, (n - 1)\}$ of integers, we will get

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

where the operations are the *usual* operations in integers. By the above calculations, we can now take remainders after division modulo n

$$(a \cdot ((b + c)\%n))\%n = ((a \cdot b)\%n + (a \cdot c)\%n)\%n$$

since taking remainder before or after arithmetic operations gives the same answer.

2. Give an example to show that addition in $\mathbb{Z}$ does not distribute over multiplication.

Solution: We need to find integers a, b and c so that

$$a + (b \cdot c) \neq (a + b) \cdot (a + c)$$

Any positive a, b, c will do. For example $a = 1$, $b = 1$, $c = 1$

$$1 + (1 \cdot 1) = 2 \neq 4 = (1 + 1) \cdot (1 + 1)$$