Characteristic Functions

1. Check the following summations as a way of verifying the formulas for characteristic functions of some discrete probability distributions.
   (a) For the Binomial distribution with $p + q = 1$
   $$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \exp(kt\sqrt{-1}) = (q + pe^{t\sqrt{-1}})^n$$
   (b) For the Poisson distribution
   $$\sum_{k=0}^{\infty} \frac{c^k}{k!} \exp(-c + kt\sqrt{-1}) = \exp\left(c\left(e^{t\sqrt{-1}} - 1\right)\right)$$
   (c) For the Negative Binomial distribution
   $$\sum_{k=0}^{\infty} \binom{n+k-1}{k} p^n q^k \exp(kt\sqrt{-1}) = \left(\frac{p}{1 - qe^{t\sqrt{-1}}}\right)^n$$

2. Check the following integrals as a way of verifying the formulas for characteristic functions of some probability densities.
   (a) For the uniform density
   $$\frac{1}{2}\int_{-1}^{1} \exp(at\sqrt{-1})da = \frac{\sin(t)}{t}$$
   Justify the limits of the integral.
   (b) For the Poisson density (exponential distribution)
   $$\int_{0}^{\infty} c\exp(-ca + at\sqrt{-1})da = \frac{c}{c - t\sqrt{-1}}$$
   where $c > 0$.
   (c) For the normal density
   $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-a^2/2 + at\sqrt{-1})da = \exp(-t^2/2)$$

3. Justify the following limits either directly or using Probability theory:
   (a) $((1 - c/n) + (c/n)e^{t\sqrt{-1}})^n$ converges to $\exp(c(e^{t\sqrt{-1}} - 1))$ as n goes to infinity.
   (b) $\cos(t/\sqrt{n})^n$ goes to $\exp(-t^2/2)$ as n goes to infinity.
   (c) If c_n goes to c as n goes to infinity, then $(1 - c_n/n)^n$ goes to e^{-c} as n goes to infinity.

4. Let X be the random variable that gives the difference of the numbers appearing when two dice are rolled. Calculate the characteristic function of X.