Solutions to Assignment 5

1. On a campus there are 100 dogs of which 20 are white.
   (a) On a certain day you spot 30 dogs what is the probability that 10 of them are white (write the formula)?
   (b) What would be the number of white dogs that you expected to see?
   (c) On another day you keep spotting dogs until you see a white dog. How many dogs would you (expect) to have spoted before you see a white dog?

Solution: The probability of a random campus dog being white is $p = 20/100 = 0.2$.

Let X be the random variable that counts the number of white dogs spotted out of $k = 30$ random campus dogs; this is a Binomial $B(k;p)$ type of random variable.

Thus, the probability that there are $r = 10$ white dogs out of $k = 30$ is

$$P(X = r) = \binom{k}{r} p^r (1-p)^{k-r} = \binom{30}{10} (0.2)^{10} (0.8)^{20}$$

The expected number of white dogs seen is $E(X) = kp = 6$.

Let W be the random variable that counts the number of non-white dogs seen before seeing a white dog; this is a Negative Binomial $NB(1;p)$ type of random variable. So $P(W = r)$ is $p(1-p)^r$. We have seen that

$$E(W) = \sum_{r=0}^{\infty} rp(1-p)^r = (1-p)/p$$

Hence, we expect to have seen 4 dogs before seeing a white dog.

2. A chemist is asked to examine a large number of samples of a product from a factory. She decides to keep examining samples until she finds 10 defective samples; at this point she has examined 150 samples (including the 10 defective ones).
   (a) How many defective samples do you expect to find in a batch of 500 samples?
   (b) If she had only looked for 3 defective samples, how many samples could she expect to have examined?
   (c) If the chemist was asked to make a box of 200 good samples how many samples would she need to examine?

Solution: The random variable W which counts the number of good samples till we find $k = 10$ defective samples is a Negative Binomial of the type $NB(k;p)$ where

p is the probability of a good sample. The expectation of W is $10p/(1-p)$, so we can *estimate* p by putting $10p/(1-p) = 150 - 10$; this gives $p = 14/15$ (which is not really surprising!).

Now, we look at the random variable X representing the the number of defective samples in a batch of 500 samples; this follows the Binomial $B(500; 1-p)$. Thus, the expected number of defective samples is $500(1-p) = 500(1/15)$ or roughly 33.

The number of good samples found while looking for 3 defects is a random variable Y distributed like $NB(3;p)$ which has expectation $3p/(1-p) = 42$. So she should expect to have looked at 45 samples in while looking for 3 defective samples.

Let T be the random variable denoting the number of defective samples found while looking for k good samples; this is a Negative Binomial Distribution $NB(k, 1-p)$. This has expectation $k(1-p)/p$. Since $p/(1-p) = 14$, we see that if $k = 200$, we have an expected $200/14$ defective samples. So we can expect to examine $200(1+1/14) = 200(15/14)$ or roughly 214 samples.

3. The police comissioner has reliable information that 10 dangerous criminals have come into the area. He sends 1000 policemen to various places in the city to look for them.
   (a) Write the formula for the probability that 5 of the policemen report back that they have seen the criminals.
   (b) The comissioner now asks a large number of citizens to help in the process. Estimate the probability that no criminal is caught.
   (c) Estimate the probability that at most 5 criminals are caught.

Solution: The probability that one policeman will spot a criminal is $p = 10/1000 = 0.01$. The probability that 5 policemen out of 1000 report back is given by the Binomal distribution and is $\binom{1000}{5}p^5(1-p)^{1000-5}$.

The probability that no policeman report back is $(1-p)^{1000}$. When we increase the number of people looking to N, the probability that any one of them spots a criminal is $10/N$. Thus, the probability that *none* of them reports back is is roughly

$$\lim_{N\to\infty}(1 - 10/N)^N = e^{-10} = .00005$$

The probability that X crimimals are caught is given by using the discrete Poisson distribution $P(X = r) = (10^r/r!)e^{-10}$. Thus the probability that at most 5 criminals are caught is

$$P(X \le 5) = e^{-10}(1 + 10 + 10^2/2 + 10^3/6 + 10^4/24 + 10^5/120)$$

or roughly 0.07.

4. Suppose that astronomers have estimated that about 10 stars in a galaxy become supernovas each year (52 weeks). A supernova can be spotted for at least a week after it explodes. An astronomer has access to a satellite that records a picture of the galaxy once every week.

   (a) How many weeks can the astronomer expect to wait before the satellite records a supernova?

   (b) Write the formula for the probability that the astronomer will spot a supernova within one month (four weeks).

   (c) The astronomer gets a grant to get him pictures every day (1 week is 7 days) from the satellite. Write the formula that the astronomer will spot the supernova within one month (28 days).

   (d) Estimate the above answers numerically. Was the effort involved in writing the grant application worth it?

Solution: The probability of a star becoming supernova in a given week is $p = 10/52$.

The random variable W that records the number of weeks of waiting required to observe a supernova has probability given by $P(W = r) = p(1-p)^r$. The expected value $E(W) = (1-p)/p = 42/10$ or roughly four weeks.

The probability that the astronomer spots a supernova within 4 weeks is:

$$P(W < 4) = p + p(1-p) + p(1-p)^2 + p(1-p)^3 = p(1-(1-p)^4)/(1-(1-p)) = 1-(1-p)^4$$

This is roughly 0.57.

One the pictures are taken every day, the probability of seeing a supernova on a given day is $q = p/7$. The random variable U that counts the number of waiting days before one spots a supernova has distribution given by $P(U = r) = q(1-q)^r$. By a calculation, similar to the one above $P(U < 28) = 1-(1-q)^{28}$ or roughly 0.54.

Strangely enough, the probability has gone down! Think of an explanation.