

The Laws of Probability

1. Which of the following are correct identities for subsets A, B, C of a fixed set S ? Provide examples if they are not correct or prove using Boole's laws if they are correct.

$$(A \cup B) - C = A \cup (B - C)$$

$$(A \cap B) \cap C = (A \cap B) \cap (C \cup B)$$

$$A \cup B = (A \setminus A \cap B) \cup B$$

$$(A \cap B) \cup (B \cap C) \cup (C \cap A) \supset A \cap B \cap C$$

$$(A \cup B)^c \cap C = (A^c \cap C) \cap (B^c \cap C)$$

$$A \cap B^c \cap C \subset A \cup B$$

2. Assume that a class has students from each part of India (North/South/East/West), comprising of Boys and Girls, so that each of the letters of the English alphabet is the starting letter for exactly one student of each sex from each region. In other words, the information (Region, Sex, Starting Letter) uniquely specifies a student. Let A be the property that a student is from the North, B be the property that the student is a Boy, C be the property that the student's name starts the letter 'C' and D be the property that the student plays basketball for IISER Mohali. Assume that a student is chosen "at random" from the class. Explain the meaning of each event (set) in the list below:

$$D^c; A \cap B; A^c \cap B^c \cap D^c; D \setminus C$$

3. Consider the set S of students in IISER Mohali. We pick a student at random. Someone asserts that the probability that the student is from Hostel 7 is 0.25, and that the probability that it is a Girl student is 0.5, and the probability that the student's name starts with 'Z' is 0.05. Finally, the person also says that the probability that is not a Girl student and not from Hostel 7 and that the name starts with A is 0.7. Is it possible that all four estimates of probability are correct? If not, why not?
4. Let Ω the set of all people in the country and E be the subset of educated people. We can think of various collections S of people such as:
- the collection of people with income less than M rupees a year.
 - the collection of people who own a cell phone.
 - the collection of people from Punjab.

For each such S , let us define $E(S) = |S \cap E|/|E|$. Check that this satisfies the axioms of a probability function on the Boolean algebra of subsets of Ω . What is an interpretation of this measure as a probability in the naive sense?

5. For each region R of the night sky, let $p(R)$ denote the percentage of visible objects that are seen in a photograph of that region. Moreover, suppose $q(R)$ is the percentage of visible galactic objects in R . Interpret $q(R)$ as a conditional probability. Suppose that 70% of all visible objects are in our galaxy. Write a formula for the (conditional) probability that an object from the region R is in our galaxy.