Quiz 10: Adjoint Functor Theorem

Question

Given a group G. Suppose $\mathcal{G}$ is the category of sets with G-action and morphisms are morphisms that commute with the G-action.

Given that F from $\mathcal{G}$ to **Set** is the forgetful functor.

1. Does F have a left adjoint? If yes, what is it?
2. Does F have a right adjoint? If yes, what is it?

Answer

If F has a left adjoint L from **Set** to $\mathcal{G}$, this would identify

$$\mathrm{Map}(T, F(S)) \text{ with } \mathcal{G}(L(T), S)$$

where $\mathrm{Map}(A, B)$ denotes the set of set-maps from a set A to a set B.

Given a set T, we have a natural action of G on the set $G \times T$ via the action on the first factor. Given a set map $a : T \to F(S)$ where S is a G-set, we have a natural map $\tilde{a} : G \times T \to S$ given by $\tilde{a}(g, t) = g \cdot a(t)$; this is a map of G-sets. Conversely, given a G-set map $b : G \times T \to S$, we have

$$b((g, t)) = b(g \cdot (e, t)) = g \cdot b(e, t)$$

for every element g in G; here e represents the identity element of G. Thus, the map b is determined by $\hat{b} : T \to F(S)$ where $\hat{b}(t) = b(e, t)$. So, if we set $L(T) = G \times T$ we have the required natural identification and L is the left-adjoint functor for F.

If F has a right adjoint R from **Set** to $\mathcal{G}$, this would identify

$$\mathrm{Map}(F(S), T) \text{ with } \mathcal{G}(S, R(T))$$

where $\mathrm{Map}(A, B)$ denotes the set of set-maps from a set A to a set B.

Given a set map $a : F(S) \to T$ where S is a G-set, we get a map $\tilde{a} : S \to \mathrm{Map}(G, T)$ given by $\tilde{a}(s)(g) = a(g \cdot s)$. Moreover,

$$\tilde{a}(h \cdot s)(g) = a(g \cdot h \cdot s) = \tilde{a}(s)(g \cdot h)$$

Note that $\mathrm{Map}(G, T)$ has a natural G action given by $(h \cdot b)(g) = b(g \cdot h)$ as above.

Conversely, given $b : S \to \mathrm{Map}(G, T)$ a map of G-sets, we define $\hat{b} : S \to T$ by $\hat{b}(s) = b(s)(e)$ and note that $b(s)(g) = b(g \cdot s)(e)$ (since b is a map of G-sets). This provides the necessary identification which shows that $R(T) = \mathrm{Map}(G, T)$ is a right adjoint to F.