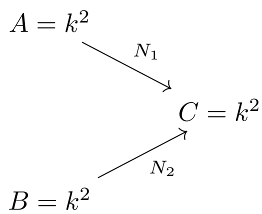
Quiz 6: Products

Question

Fix a field k.

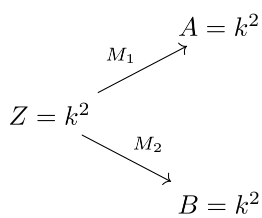
Consider the category $\mathbf{Vect}_k$ with objects vector spaces over k and morphisms as k-linear maps between vector spaces.

Consider the diagram D given by



where $N_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $N_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

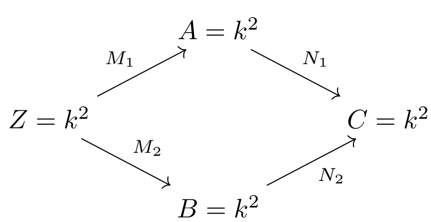
Find suitable matrices M_1 and M_2 so that if $Z = k^2$ then



gives the product of the diagram D in the category $\mathbf{Vect}_k$.

Answer

Note that the *first* requirement is that $N_1 M_1 = N_2 M_2$ so that the following diagram commutes:



This shows that (N_1, N_2) gives a morphism from Z to the diagram D.

What we are asking is that (Z, N_1, N_2) be the fibre-product $A \times_C B$ in the category $\mathbf{Vect}_k$.

One useful observation is that since $N_1 : A \to C$ is an isomorphism, we can conclude that $M_2 : A \times_C B \to B$ is an isomorphism. Since, N_2 is also an isomorphism, we see that $N_2 \circ M_2 : A \times_C B \to C$ is an isomorphism.

It follows that $A \times_C B$ is isomorphic to C under $N_2 \circ M_2 = N_1 \circ M_1$. Hence, we set $Z = k^2$ as was done in the question. In fact, we can take $N_2 \circ M_2 = N_1 \circ M_1$ to be *identity* since the fibre-product is determined upto isomorphism.

Thus, it is enough to take

$$M_1 = N_1^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } M_2 = N_2^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

More generally, we can take $M_1' = M_1 K$ and $M_2' = M_2 K$ for *any* invertible matrix K and M_1 and M_2 as above.