Universal objects

We have defined the *final object* of a category $\mathcal{C}$ as an object 1 in $\mathcal{C}$ such that for every object A of $\mathcal{C}$, there is a unique morphism $A \to 1$.

Similarly, the *initial object* of a category $\mathcal{C}$ is an object 0 in $\mathcal{C}$ such that for every object A of $\mathcal{C}$, there is a unique morphism $0 \to A$.

Both of these were special cases of products and co-products of diagrams.

We now turn this around to see that initial and final objects of certain categories give rise to products and other constructions in category theory.

In order to do this, we need to *construct* various categories.

Diagrams via categories and functors

Given a directed graph Γ, we can construct a category $\mathbf{P}_\Gamma$ as follows:

- The objects are the vertices of Γ.
- The morphisms are finite directed paths in Γ; in other words, given vertices v and w, a morphism $v \to w$ is a sequence $(e_1, \ldots, e_n)$ of edges in Γ such that $l(e_1) = v$, $r(e_i) = l(e_{i+1})$ for all i and $r(e_n) = w$.
- By convention, the *empty* sequence can be thought of as the identity morphism $1_v : v \to v$.
- Conposition of morphisms is the composition of paths in the usual way.

One easily checks that the identity and associativity of composition holds, so $\mathbf{P}_\Gamma$ is a category.

Recall that if D is a diagram in a category $\mathcal{C}$ associated with the directed graph Γ:

- Each vertex v in Γ corresponds to an object A_v in $\mathcal{C}$.
- Each directed edge e in Γ that goes from v to w is associated with a morphism $f_e : A_v \to A_w$.

This gives us a *functor* $F_D : \mathbf{P}_\Gamma \to \mathcal{C}$ which sends v to A_v and $(e_1, \ldots, e_n)$ to the composition $f_{e_n} \circ \cdots f_{e_1}$. Conversely, given a functor $F : \mathbf{P}_\Gamma \to \mathcal{C}$, we have the diagram D_F:

- A vertex v in Γ is associated to the object $F(v)$ in $\mathcal{C}$.
- An edge e from v to w in Γ gives a morphism $(e) : v \to w$ in $\mathbf{P}_\Gamma$ and thus a morphism $F(e) : F(v) \to F(w)$ in $\mathcal{C}$.

Thus, we see that giving a diagram D based on the directed graph Γ is the same as giving a functor $F : \mathbf{P}_\Gamma \to \mathcal{D}$.

A trivial, but important, example

Given an object A in a category $\mathcal{C}$ and *any* category $\mathcal{D}$, there is a functor $\mathcal{D} \to \mathcal{C}$ that associates, to every object of $\mathcal{D}$, the object A, and to every morphism of $\mathcal{D}$, the identity morphism 1_A. Let us denote this functor by ΔA.

Given a directed graph Γ, we can associate the diagram in $\mathcal{C}$ such that:

- For a vertex v of Γ, the associated object is A.
- For an edge e of Γ, the associated morphism in 1_A.

This corresponds precisely to the above functor $\Delta A : \mathbf{P}_\Gamma \to \mathcal{C}$.

Morphisms to/from a diagram

We had introduced the notion of a morphism from an object X in $\mathcal{C}$ to a diagram D in $\mathcal{C}$:

- For each vertex v in D, we have a morphism $x_v : X \to A_v$ in $\mathcal{C}$.
- For each edge e from v to w, we have $x_w = f_e \circ x_v$ where $f_e : X_v \to X_w$ is the morphism in $\mathcal{C}$ associated with e.

Now, consider the functor $F_D : \mathbf{P}_\Gamma \to \mathcal{C}$ associated with the diagram D and the functor $\Delta X : \mathbf{P}_\Gamma \to \mathcal{C}$.

- For each vertex v we have a morphism $x_v : X(v) = X \to F_D(v) = A_v$.
- For each edge e we have a commutative diagram

$$\begin{array}{ccc} X & \overset{X(e)=1_A}{=\!=\!=\!=} & X \\ {\scriptstyle x_v}\big\downarrow & & \big\downarrow{\scriptstyle x_w} \\ A_v & \xrightarrow[f_e=F_D(e)]{} & A_w \end{array}$$

More generally, for a morphism $(e_1, \ldots, e_n)$ in $\mathbf{P}_\Gamma$ we get a "ladder" of commutative diagrams

$$\begin{array}{ccccccc} X & =\!=\!=\!= & X & \cdots\cdots & X & =\!=\!=\!= & X \\ {\scriptstyle x_v}\big\downarrow & & \big\downarrow{\scriptstyle x_{r(e_1)}} & & \big\downarrow{\scriptstyle x_{l(e_n)}} & & \big\downarrow{\scriptstyle x_w} \\ A_v & \xrightarrow[f_{e_1}]{} & A_{r(e_1)} & \cdots\cdots & A_{l(e_n)} & \xrightarrow{f_{e_n}} & A_w \end{array}$$

This shows that we have a *natural transformation* $\chi : \Delta X \to F_D$ of functors which associates to each object v of $\mathbf{P}_\Gamma$ the morphism $x_v : X \to A_v$. Conversely, we see easily that natural transformation $\Delta X \to F_D$ gives a morphism from X to the diagram D of $\mathcal{C}$ as defined earlier.

Similarly, a morphism from a diagram D to an object Z in $\mathcal{C}$ can be seen as a natural transformation $\xi : F_D \to \Delta Z$ of functors from $\mathbf{P}_\Gamma$ to $\mathcal{C}$.

Products as final objects

We can summarise the above two subsections as follows. We can replace the study of diagrams D in a category $\mathcal{C}$ by the study of functors F_D from $\mathbf{P}_\Gamma$ to $\mathcal{C}$.

Moreover, the study of morphisms from an object X of $\mathcal{C}$ to the diagram D can be replaced by the study of natural transformations $\Delta X \to F_D$.

We now consider the category $\mathcal{C}_D$ as follows:

- Objects are pairs (X, χ) where X is an object of $\mathcal{C}$ and $\chi : \Delta X \to F_D$ is a natural transformation of functors.
- A morphism $f : (X, \chi) \to (Y, \eta)$ is given by a morphism $f : X \to Y$ such that the following diagram of natural transformations commutes

$$\begin{array}{ccc} \Delta X & \xrightarrow{\chi} & F_D \\ {\scriptstyle \Delta f}\downarrow & & \| \\ \Delta Y & \xrightarrow{\eta} & F_D \end{array}$$

  Here Δf denotes the natural transformation $\Delta X \to \Delta Y$ which is $f : X \to Y$ for each vertex of Γ.

The definition of the product $\prod D$ of the diagram now easily translates into the assertion that $\prod D$ is a final object of the category $\mathcal{C}_D$.

Co-products as initial objects

Similarly, we can use the fact that morphisms from the diagram D to an object Z of $\mathcal{C}$ can be replaced natural transformations $F_D \to \Delta Z$ to reformulate co-products.

We now consider the category $\mathcal{C}^D$ as follows:

- Objects are pairs (Z, ξ) where Z is an object of $\mathcal{C}$ and $\xi : F_D \to \Delta Z$ is a natural transformation of functors.
- A morphism $f : (Z, \xi) \to (W, \omega)$ is given by a morphism $f : Z \to W$ such that the following diagram of natural transformations commutes

$$\begin{array}{ccc} F_D & \xrightarrow{\xi} & \Delta Z \\ \| & & {\scriptstyle \Delta f}\downarrow \\ F_D & \xrightarrow{\omega} & \Delta W \end{array}$$

The definition of the co-product $\coprod D$ of the diagram now easily translates into the assertion that $\coprod D$ is a initial object of the category $\mathcal{C}^D$.

Adjoints

As part of the philosophy that various constructions in category theory can be formulated as finding initial and final objects of suitable categories, we now look at the problem of constructing adjoint functors.

Given a functor $F : \mathcal{C} \to \mathcal{D}$, and an object D of $\mathcal{D}$ we define a category $D \downarrow F$ as follows:

- Objects are pairs (A, h) where A is an object of $\mathcal{C}$ and $h : D \to FA$ is a morphism in $\mathcal{D}$.
- A morphism $f : (A, h) \to (B, g)$ is given by a morphism $f : A \to B$ in $\mathcal{C}$ such that the following diagram of morphisms in $\mathcal{D}$ commutes

$$\begin{array}{ccc} D & \xrightarrow{h} & FA \\ \Vert & & \downarrow{\scriptstyle Ff} \\ D & \xrightarrow{g} & FB \end{array}$$

Note that if G is a *left adjoint* of F, then GD is an object of $\mathcal{C}$ such that $\mathcal{C}(GD, A) = \mathcal{D}(D, FA)$. In other words each object (A, h) of $D \downarrow F$ corresponds *uniquely* to a morphism $f_h : GD \to A$ in $\mathcal{C}$. Moreover, we have the unit morphism $u_D : D \to FGD$ of the adjunction for which we have $h = F(f_h) \circ u_D$. So we have a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{u_D} & FGD \\ \Vert & & \downarrow{\scriptstyle F(f_h)} \\ D & \xrightarrow{h} & FA \end{array}$$

In other words, f_h gives a morphism in $D \downarrow F$ as well. It follows easily that GD is an initial object in this category.

If an initial object (C, u) of $D \downarrow F$ exists, it is *like* (GD, u_D), where G is a left adjoint of F.

- Given an object (A, h) in $D \downarrow F$, there is a unique morphism $f_h : C \to A$ in $\mathcal{C}$ such that $h : D \to FA$ is the composite $h = F(f_h) \circ u$.
- Conversely, given a morphism $f : C \to A$ in $\mathcal{C}$ we note that $(A, F(f) \circ u)$ is an object in $D \downarrow F$.

So, we see that $\mathcal{C}(C, A) = \mathcal{D}(D, FA)$.

Right adjoint

Similarly, given an object D of $\mathcal{D}$, we consider the category $F \downarrow D$ as follows:

- Objects are pairs (A, h) where A is an object of $\mathcal{C}$ and $h : FA \to D$ is a morphism in $\mathcal{D}$.
- A morphism $f : (A, h) \to (B, g)$ is given by a morphism $f : A \to B$ in $\mathcal{C}$ such that the following diagram of morphisms in $\mathcal{D}$ commutes

$$\begin{array}{ccc} FA & \xrightarrow{h} & D \\ \downarrow{\scriptstyle Ff} & & \Vert \\ FB & \xrightarrow{g} & D \end{array}$$

Dualising the arguments above, we see that if a final object (C, v) of $F \downarrow D$ exists, it is *like* (GD, v_D), where G is a right adjoint of F.

- Given an object (A, h) in $F \downarrow D$, there is a unique morphism $f^h : A \to C$ in $\mathcal{C}$ such that $h : FA \to D$ is the composite $h = v \circ F(f^h)$.
- Conversely, given a morphism $f : A \to C$ in $\mathcal{C}$ we note that $(A, v \circ F(f))$ is an object in $F \downarrow D$.

So, we see that $\mathcal{C}(A, C) = \mathcal{D}(FA, D)$.