Some special types of morphisms

For an object A of a category $\mathcal{C}$, we have seen that morphisms $x : X \to A$ can be seen as "elements of A of type X". This led us to the functor from $\mathcal{C}$ to the category $\mathbf{Fun}(\mathcal{C}^{\mathrm{opp}}, \mathbf{Set})$ of contravariant functors: an object A is associated with the functor $A^{\cdot}$ where $A^{\cdot}(X) = \mathcal{C}(X, A)$. We saw that this functor is full and faithful. In fact, for a contravariant functor F from $\mathcal{C}$ to **Set**, we have a natural identification between elements of $F(A)$ and natural transformations $A^{\cdot} \to F$.

This leads us to the study of special types of morphisms inside $\mathcal{C}(A, B)$.

Monic morphisms

A morphism $f : A \to B$ is said to be *monic* or a *monomorphism* if the resulting set map $A^{\cdot}(X) \to B^{\cdot}(X)$ is one-to-one for every object X of $\mathcal{C}$.

Put differently, given an object X of $\mathcal{C}$ and morphisms $a_i : X \to A$ for $i = 1, 2$, we have:

$$f \circ a_1 = f \circ a_2 \text{ if and only if } a_1 = a_2$$

A morphism $f : A \to B$ in the category **Set** (i.e. a set map) is *determined* by the set map $A^{\cdot}(X) \to B^{\cdot}(X)$, where $X = \{\cdot\}$ denotes the singleton set. It follows that a morphism in **Set** is monic if and only if it is one-to-one in the usual sense.

The same argument can be applied to **Top**, by taking $X = \{\cdot\}$ to be the singleton *space*.

In the category **Grp**, we take $X = \mathbb{Z}$, the group of integers. One sees that a group G is *determined* by the elements $G^{\cdot}(X) = \mathrm{Hom}(\mathbb{Z}, G)$. It follows that a morphism in **Set** is monic if and only if it is one-to-one in the usual sense.

The same argument can be applied to $\mathbf{Vect}_k$ of vector spaces over a field k by taking $X = \{\cdot\}^1$, the standard 1-dimensional vector space over k.

More generally, if F is a faithful functor from a category $\mathcal{C}$ to the category **Set**, and $f : A \to B$ is a morphism in $\mathcal{C}$ such that $F(f) : F(A) \to F(B)$ is one-to-one, then f is monic. To see this, let $a_i : X \to A$ for $i = 1, 2$ be morphisms in $\mathcal{C}$ such that $f \circ a_1 = f \circ a_2$. It follows that

$$F(f) \circ F(a_1) = F(f \circ a_1) = F(f \circ a_2) = F(f) \circ F(a_2)$$

Since $F(f)$ is one-to-one and thus monic in **Set**, it follows that $F(a_1) = F(a_2) : F(X) \to F(A)$. Using the fact that F is *faithful*, it follows that $a_1 = a_2$.

Note that this *does not* say that monic morphisms in $\mathcal{C}$ are *precisely* those such that their image under F is one-to-one.

For example, consider the category $\mathbf{1}_{\to}$ (introduced earlier) which has a unique non-identity morphism $f : A \to B$ between two distinct objects A and B. In this category, f is clearly monic.

It is also clear that *any* functor from $\mathbf{1}_{\rightarrow}$ to **Set** which takes f to a morphism between distinct objects is faithful; this follows since the three morphisms in $\mathbf{1}_{\rightarrow}$ go to distinct morphisms in **Set**. In particular, we can take A to the set $\{0,1\}$ and B to the set $\{0\}$ and f to the unique set map $\{0,1\} \rightarrow \{0\}$. Thus, the image of f is *not* one-to-one even though f is monic.

Epic morphisms

We *dualise* the above notion by reversing all the arrows.

A morphism $f : A \rightarrow B$ is said to be *epic* or an *epimorphism* if, given an object X of $\mathcal{C}$ and morphisms $b_i : B \rightarrow X$ for $i = 1, 2$, we have:

$$b_1 \circ f = b_2 \circ f \text{ if and only if } b_1 = b_2$$

Given a subset S of a set A, we have a map $\chi_S : A \rightarrow \{0,1\}$ which maps S to $\{0\}$ and the complement of S to $\{1\}$. Moreover, for subsets S and T of A, we have $\chi_S = \chi_T$ if and only if $S = T$.

Now consider a map $f : B \rightarrow A$ whose image is S. The image of $\chi_S \circ f$ is $\{0\}$ which is the same as the image of $\chi_A \circ f$. It follows that $\chi_S \circ f = \chi_A \circ f$. Thus, we see that f is an epimorphism if and only if $S = A$. In other words, epic morphisms in **Set** are *precisely* set maps which are onto.

Let $D \hookrightarrow X$ be a dense subset of a topological space X. Two continuous maps $a, b : X \rightarrow Y$ are equal if the restrictions $a_{|_D}, b_{|_D}$ are equal. It follows that $f : Z \rightarrow X$ is epic if the image $f(Z)$ is a dense subset of X. In particular, an epic morphism *need* not be onto.

Given a subgroup $B \hookrightarrow A$ of an Abelian group, we can take $X = A/B$. The natural homomorphism $q : A \rightarrow A/B = X$ and the homomorphism $0 : A \rightarrow X$ become equal when restricted to B. Given a homomorphism $f : C \rightarrow A$, let $B = f(C)$. It follows that the homomorphism $f : C \rightarrow A$ is epic if and only if $f(C) = A$. Note that this argument does *not* work when A is not Abelian since B need not be a normal subgroup.

More generally, if F is a faithful functor from a category $\mathcal{C}$ to the category **Set**, and $f : A \rightarrow B$ is a morphism in $\mathcal{C}$ such that $F(f) : F(A) \rightarrow F(B)$ is onto, then f is epic. To see this, let $b_i : B \rightarrow X$ for $i = 1, 2$ be morphisms in $\mathcal{C}$ such that $b_1 \circ f = b_2 \circ f$. It follows that

$$F(b_1) \circ F(f) = F(b_1 \circ f) = F(b_2 \circ f) = F(b_2) \circ F(f)$$

Since $F(f)$ is onto and thus epic in **Set**, it follows that $F(b_1) = F(b_2) : F(B) \rightarrow F(X)$. Using the fact that F is *faithful*, it follows that $b_1 = b_2$.

Note that this *does not* say that epic morphisms in $\mathcal{C}$ are *precisely* those such that their image under F is onto.

For example, we can consider the category $\mathbf{1}_\to$ (introduced earlier) which has a unique non-identity morphism $f : A \to B$ between two distinct objects A and B. In this category, f is clearly epic.

Analogously to the earlier discussion, we can consider the functor from $\mathbf{1}_\to$ to **Set** that associates the set $\{0\}$ to A and the set $\{0, 1\}$ to B with the natural inclusion of $\{0\}$ to $\{0, 1\}$ associated with the morphism f. As seen above, this functor is faithful. Yet, the image of f is *not* onto even though f is epic.

Sections

In analogy with the case of monic morphisms, we could also study morphisms $f : A \to B$ such that the resulting set map $A^\cdot(X) \to B^\cdot(X)$ is onto for all objects X of $\mathcal{C}$.

For such a morphism f, we get an onto map $A^\cdot(B) \to B^\cdot(B)$. Hence, there is an element s of $A^\cdot(B) = \mathcal{C}(B, A)$ such that $f \circ s = 1_B$ is the identity element of $B^\cdot(B) = \mathcal{C}(B, B)$.

Given a morphism $f : A \to B$, we say that $s : B \to A$ is a *section* of f if $f \circ s = 1_B$; alternatively, we also say that f *has a section* s.

So we see that if $A^\cdot(X) \to B^\cdot(X)$ is onto for all objects X of $\mathcal{C}$, then $f : A \to B$ has a section.

Conversely, suppose that $f : A \to B$ has a section $s : B \to A$. Given any element x in $B^\cdot(X) = \mathcal{C}(X, B)$, we have the element $s \circ x$ in $A^\cdot(X) = \mathcal{C}(X, A)$ which satisfies $f \circ (s \circ x) = (f \circ s) \circ x = x$. Thus, $A^\cdot(X) \to B^\cdot(X)$ is onto.

Given a morphism $f : A \to B$, suppose $b_1 : B \to X$ and $b_2 : B \to X$ are two morphisms such that $b_1 \circ f = b_2 \circ f$. If f has a section $s : B \to A$, then composing this identity with s we obtain

$$b_1 = b_1 \circ (f \circ s) = (b_1 \circ f) \circ s = (b_2 \circ f) \circ s = b_2 \circ (f \circ s) = b_2$$

It follows that if f has a section, then f is epic.

Retractions

Dualising the above, we could study morphisms $f : A \to B$ such that there is a morphism $r : B \to A$ such that $r \circ f = 1_A$. In this case, we say that f has a *retraction* r, or that r is a *retraction* of f.

Given a retraction r of f, consider two elements $a_i : X \to A$ for $i = 1, 2$ such that $f \circ a_1 = f \circ a_2$. Further composing with r, we obtain

$$a_1 = (r \circ f) \circ a_1 = r \circ (f \circ a_1) = r \circ (f \circ a_2) = (r \circ f) \circ a_2 = a_2$$

Thus, we see that if f has a retraction r, then f is monic.

Examples

We give examples from algebra and topology in the above context.

In the category **Grp** of groups, consider the natural onto homorphism $\mathbb{Z}/4 \to \mathbb{Z}/2$. As seen above, this is an epimorphism. However, it does not have a section.

Similarly, consider the natural one-to-one homomorphism $\mathbb{Z}/2 \to \mathbb{Z}/4$. As seen above, this is an epimorphisms. However, it does not have a retraction.

By the usual argument of countability of rational numbers, there is a one-to-one onto map $r : \mathbb{N} \to \mathbf{Q}^+$ from the counting numbers $\mathbb{N}$ to the positive rational numbers $\mathbb{Q}^+$. If we equip $\mathbb{N}$ with the discrete topology, then this map is continuous *whatever* topology we given to $\mathbb{Q}^+$; let us give $\mathbb{Q}^+$, the *usual* topology coming from ordering of rational numbers. We thus consider the map r as a morphism in the category **Top** of topological spaces. As seen above, this morphism is *both* monic as well as epic!

If $s : \mathbb{Q}^+ \to \mathbb{N}$ is to be a section or retraction of r, then $s = r^{-1}$ since r is a bijection. However, it is clear that r^{-1} is *not* continuous; which means that it is *not* a morphism in **Top**. In other words, r is a monic and epic morphism in **Top** that neither has a section nor a retraction.