Monad of Groups

In this section we will demonstrate a monad F the “creates” the theory of groups as F-algbras.

Recall that a monad (F, u, m) on **Set** is:

- A functor F from **Set** to itself.
- A natural transformation $u : 1_{\mathbf{Set}} \to F$.
- A natural transformation $m : FF \to F$.
- The following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{u_F} & FF \\ {\scriptstyle F(u)}\big\downarrow & \overset{1_F}{\searrow\!\!\searrow} & \big\downarrow{\scriptstyle m} \\ FF & \xrightarrow{m} & F \end{array}$$

- The following diagram commutes:

$$\begin{array}{ccc} FFF & \xrightarrow{F(m)} & FF \\ \big\downarrow{\scriptstyle m_F} & & \big\downarrow{\scriptstyle m} \\ FF & \xrightarrow{m} & F \end{array}$$

Recall that an M-algebra is a set H with a map $\mu : M(H) \to H$ which satisfies:

- $\mu \circ u_H : H \to H$ is the identity map.
- $\mu \circ F(m_H) = \mu \circ F(\mu)$ as set maps $FF(H) \to H$.

The free group functor F

To a set S we associate the set $F(S)$ whose elements are (conceptually) products of variables $(g_a)_{a \in S}$ and their inverses. In other words, an element w of $F(S)$ has the form

$$w = g_{a_1}^{n_1} \cdots g_{a_r}^{n_r}$$

where a_i are elements of S and $n_i \in \mathbb{Z}$ are integers. To avoid redundancy, we require that a_i is *different* from a_{i+1} and do not allow $n_i = 0$. Finally, we allow $r = 0$; this corresponds to the *empty* product which we denote as e. We call such elements w of $F(S)$ “words” in the “group generators” S.

Given a more general word *with* possible redundancy (i.e. where we allow $a_i = a_{i+1}$), we can *shrink* the the word by *combining* $g_a^n g_a^m$ to g_a^{m+n}. When $m + n = 0$ we drop this term entirely. Each such step *reduces* the length of the expression and we denote this operation by $\rho(w)$ and call the result the *reduced* word.

Given a set map $f : S \to T$, we get a set map $F(f) : F(S) \to F(T)$ by defining

$$F(f)(w) = F(f)(g_{a_1}^{n_1} \cdots g_{a_r}^{n_r}) = \rho\left(g_{f(a_1)}^{n_1} \cdots g_{f(a_r)}^{n_r}\right)$$

Note that the reduction ρ is used since $f(a_i)$ *could* be equal to $f(a_{i+1})$.

One easily checks that F defines a functor from **Set** to itself.

The natural transformations u and m

We define $u_S : S \to F(S)$ by $a \mapsto g_a$. It is clear that this gives a natural transformation $1_{\mathbf{Set}} \to F$.

An element of $FF(S)$ can be written in the form $g_{w_1}^{n_1} \cdots g_{w_r}^{n_r}$ where n_i are non-zero integers and $w_i \in F(S)$ are words such that $w_i \neq w_{i+1}$.

Under the map $m_S : FF(S) \to M(S)$

$$m_S : g_{w_1}^{n_1} \cdots g_{w_r}^{n_r} \mapsto \rho\left(w_1^{n_1} \cdots w_r^{n_r}\right)$$

Note that when w_i is the empty word, it is dropped (actually, it does not appear!) from the expression above as part of the reduction. This remark is necessary since g_e is *could* occur as g_{w_i} for some i.

One can check the commutative diagrams which show that (F, u, m) is a monad (but it will not be done in these notes and left as an exercise for the reader).

F-algebras

Given a morphism $\mu : F(H) \to H$ together with the commudative diagrams which make this an F-alebgra, we wish to show that H becomes a *group* in a natural way so that the map μ is given by

$$\mu : g_{a_1}^{n_1} \cdots g_{a_r}^{n_r} \mapsto a_1^{n_1} \odot \cdots \odot a_r^{n_r}$$

where:

- for a in H and n an integer, a^n is the n-th power of the group element $a \in H$, and
- for a and b in H, $a \odot b$ represents the product of these elements in H.

To prove this, we first *define* the various components of the proposed group structure on H as follows.

- The element $\mu(e)$ is denoted by 1 and will be shown to be the identity element of H.
- The element $\mu(g_a^2)$ is denoted by $a \odot a$ and will be shown to be the product of a with itself in H.
- When a, b are different, the element $\mu(g_a g_b)$ is denoted by $a \odot b$ and will be shown to be the product in H.
- The element $\mu(g_a^{-1})$ is denoted by $\iota(a)$ and will be shown to be the inverse of the element a in H.

The identity $\mu(g_a) = a$

The first condition for F-algebras says that $\mu(u_H(a)) = a$. Since $u_H(a) = g_a$, we see that we obtain $\mu(g_a) = a$ as required.

In particular, we note that $\mu(g_1) = 1 = \mu(e)$ and $\mu(g_{\iota(a)}) = \iota(a) = \mu(g_a^{-1})$. Similarly, we see that $\mu(g_{a\odot b}) = a \odot b = \mu(g_a g_b)$. Thus, the map μ can take different elements in $F(H)$ to the same element in H.

A formula

Before going further, let us compute $F(\mu) : FF(H) \to F(H)$.

$$F(\mu) : g_{w_1}^{n_1} \cdots g_{w_r}^{n_r} \mapsto \rho\left(g_{\mu(w_1)}^{n_1} \cdots g_{\mu(w_r)}^{n_r}\right)$$

Now using the identity $\mu \circ F(\mu) = \mu \circ m_H$ and the computation of m_H above we see that

$$\mu\left(\rho\left(w_1^{n_1} \cdots w_r^{n_r}\right)\right) = \mu\left(\rho\left(g_{\mu(w_1)}^{n_1} \cdots g_{\mu(w_r)}^{n_r}\right)\right)$$

for all choices of w_i and n_i as above.

The element 1 is identity for $\odot$

We apply the above identity with $w_1 = g_a$ and $w_2 = e$ to get

$$\mu\left(\rho\left(g_a e\right)\right) = \mu\left(\rho\left(g_{\mu(g_a)} g_{\mu(e)}\right)\right)$$

Now the left-hand side simplifies to $\mu(g_a) = a$, while the right-hand side simplifies to $\mu(g_a g_1) = a \odot 1$. This shows that $a = a \odot 1$.

Similarly, if we take $w_1 = e$ and $w_2 = g_a$, we get $a = 1 \odot a$.

The element $\iota(a)$ is the inverse of a

We apply the above identity with $w_1 = g_a$ and $w_2 = g_a^{-1}$ to get

$$\mu\left(\rho\left(g_a g_a^{-1}\right)\right) = \mu\left(\rho\left(g_{\mu(g_a)} g_{\mu(g_a^{-1})}\right)\right)$$

Now the left-hand side simplifies to $\mu(e) = 1$, while the right-hand side simplifies to $\mu(g_a g_{\iota(a)}) = a \odot \iota(a)$. This shows that $1 = a \odot \iota(a)$.

Similarly, if we take $w_1 = g_a^{-1}$ and $w_2 = g_a$, we get $1 = \iota(a) \odot a$.

Associativity

We apply the above identity with $w_1 = g_a$ and $w_2 = g_b g_c$ to get

$$\mu\left(\rho\left(g_a g_b g_c\right)\right) = \mu\left(\rho\left(g_{\mu(g_a)} g_{\mu(g_b g_c)}\right)\right)$$

Now the left-hand side simplifies to $\mu(g_a g_b g_c)$, while the right-hand side simplifies to $\mu(g_a g_{b\odot c}) = a \odot (b \odot c)$. This shows that $a \odot (b \odot c) = \mu(g_a g_b g_c)$

Similarly, if we take $w_1 = g_a g_b$ and $w_2 = g_c$, we get $(a \odot b) \odot c = \mu(g_a g_b g_c)$. Combining these identities gives $(a \odot b) \odot c = a \odot (b \odot c)$ as required.

It follows that H is a group with the above operations. The computation of μ in terms of the group operation also follows rather easily after that.