Power Sets and Lattices

Recall that:

- We have a functor P from **Set** to itself that associates to a set S, the power set $P(S)$ that parametrises subsets of S.
- To a set map $f : S \to T$, the associated set map $P(f) : P(S) \to P(T)$ is the map that takes a subset R of S to its image $f(R)$ in T.
- The natural transformation u given by $u_S : S \to P(S)$ takes an element s of S to the singleton set $\{s\}$ which is an element of $P(S)$.
- A subset T of $P(S)$ gives an indexed collection $\{R_t\}_{t\in T}$ of subsets R_t of S for each t. We define $m(T) = \cup_{t\in T} R_t$ which is the union of the R_t as subsets of S. This defines a set map $m_S : P(P(S)) \to P(S)$ which is a natural transformation $m : PP \to P$.

Together the above data defines a *monad* on the category **Set**.

Recall that a P-algebra is a set L with a map $v : P(L) \to L$ which satisfies:

- $v \circ u_L : L \to L$ is the identity map.
- $v \circ m_L$ and $v \circ P(v)$ define the *same* morphism $PP(L) \to L$.

In the following discussion we will try to give a more "traditional" set-theoretic meaning to this notion. Specifically, we will prove that a P-algebra is a "bounded sup Lattice"; the latter term will be explained as we go along.

Partial order on a P-algebra

We begin by noting that $u_L(a) = \{a\}$ for each element of L. Hence, we see that the first condition above gives us $v(\{a\}) = a$.

Let us define the following notions:

- Define $0 = v(\emptyset)$ to be the image under v of the empty set $\emptyset$ as an element of $P(L)$.
- Define $1 = v(L)$ to be the image under v of the whole set L as an element of $P(L)$.
- Define $a \leq b$ if $v(\{a, b\}) = b$.

We will first prove that $\leq$ is a partial order on L, with 0 as the smallest element and 1 as the largest element.

First of all note that if a is an element of L, then the image of $\{\emptyset, \{a\}\}$ under m_L is $\{a\}$ in $P(L)$. On the other hand ts image under $P(v)$ is $\{0, a\}$. The second condition says that $v(\{0, a\}) = v(\{a\}) = a$ so it follows that $0 \leq a$ for every a in L.

Next, we note that the image of $\{\{a\}, L\}$ under m_L is L as an element in $P(L)$, while the image under $P(v)$ is $\{a, 1\}$. By the second condition $v(\{a, 1\}) = v(L) = 1$, so we see that $a \leq 1$ for every a in L.

We see that $v(\{a,a\}) = v(\{a\}) = a$ which shows that $a \leq a$. Clearly, if $a \leq b$ and $b \leq a$, then $b = v(\{a,b\}) = a$.

Given that $v(\{a,b\}) = b$ and $v(\{b,c\}) = c$, consider the set $\{\{a\},\{b,c\}\}$. Its image under m_L is $\{a,b,c\}$ while its image under $P(v)$ is $\{a,c\}$. Thus, we see that $v(\{a,c\}) = v(\{a,b,c\})$. On the other hand consider the set $\{\{a,b\},\{c\}\}$. Its image under m_L is $\{a,b,c\}$ while its image under $P(v)$ is $\{b,c\}$. Thus, we see that $v(\{a,b,c\}) = v(\{b,c\}) = c$. It follows that $a \leq c$ as required.

Hence, we have proved that $\leq$ is a partial order on L such that 0 is the minimum and 1 is the maximum.

$v(S)$ as the least upper bound

Given subsets S and T of L such that $S \subset T$, consider the element $\{S,T\}$ of $PP(L)$. Its image under m_L is $S \cup T = T$ while its image under $P(v)$ is $\{v(S), v(T)\}$. Hence, by the second condition, it follows that $v(\{v(S), v(T)\}) = v(T)$ which shows that $v(S) \leq v(T)$.

Suppose S is a subset of L and a lies in S. Then $\{a\}$ is a subset of S so it follows that $a = v(\{a\}) \leq v(S)$. So $v(S)$ is an upper bound of the elements of S.

Suppose S is a subset of L and a is an upper bound of S. This means that for every b in S, we have $b \leq a$ so that $v(\{a,b\}) = a$. Now, consider the element $\{\{a,b\} : b \in S\}$ in $PP(L)$. Its image under m_L is $S \cup \{a\}$, while its image under $P(v)$ is $\{a\}$. It follows that $v(S \cup \{a\}) = a$ and thus, $v(S) \leq v(S \cup \{a\}) = a$.

Hence $v(S)$ is the *least* upper bound of the set S in L.

Bounded sup lattice

Hence, we see that L is a partially ordered set such that every subset S has a least upper bound $v(S)$. Moreover, L has a least element $0 = v(\emptyset)$ and a greatest element $1 = v(L)$.

Now, if S is a subset of L, then let S' denote the subset which consists of lower bounds for S. Each b in S is an upper bound for elements of S'. Since $v(S')$ is a least upper bound for S' we get $v(S') \leq b$. Since this happens for each b in S, we see that that $v(S')$ is also a lower bound of S. Hence, every set S also has a greatest lower bound which is $v(S')$. In particular, given any pair a, b of elements we have an element $a \wedge b$ which is their greatest lower bound.

Alternative terminology for least upper bound is *supremum* and for greatest lower bound is *infimum*.

A "bounded lattice" is a partially ordered set L which has:

- A supremum $a \vee b$ of a pair of elements a, b.
- An infimum $a \wedge b$ of a pair of elements a, b.
- A least element 0.
- A greatest element 1.

A “bounded sup lattice” is a bounded lattice that also has a supremum for *every* subset S of L.

The reason for emphasizing the role of supremum in the name is that the supremum map v occurs in the *definition* of the P-algebra L. As we shall see, this will be important when we look at morphisms of p-algebras; these will be *required* to preserve supremum, but may not preserve infimum.