Quiz 1: Examples of Categories

Question: Describe all (if any) categories that have exactly one object and exactly one morphism.

Answer: We easily verify the following statements:

1. Let the unique object be denoted as A. All morphisms are from A to itself.
2. One of the morphisms *must* be the identity morphism 1_A. Let f be the other morphism.
3. We have the identities $1_A \circ 1_A = 1_A$, $1_A \circ f = f \circ 1_A = f$. Thus, the only thing left to determine is $f \circ f$.
4. Either $f \circ f = 1_A$ or $f \circ f = f$. Let us now check the associative law in each case.

   $f \circ f = 1_A$: In this case f is an isomorphism and of order 2. Thus, the collection of morphisms forms the group isomorphic to the (additive) cyclic group $\mathbb{Z}/\langle 2 \rangle = \{0, 1\}$ with 0 playing the role of 1_A and 1 playing the role of f.

   $f \circ f = f$: We check that $f \circ (f \circ f) = f \circ f = (f \circ f) \circ f$; similarly, the other identities for the associative law. Alternatively, we can think of the *multiplicative* monad $\{0, 1\}$. with 0 playing the role of f and 1 playing the role of 1_A.