Categories as sets with structure

In this section we will be interested in categories where the objects and morphisms form sets.

Small category

A *small* category $\mathcal{C}$ is given by a tuple (C_0, C_1, r, l, i, μ) where:

- C_0 and C_1 are sets which are thought of as the set of objects and the set of morphisms, respectively.
- $l : C_1 \to C_0$ and $r : C_1 \to C_0$ are set maps where a morphism $f \in C_1$ is thought of as an arrow $l(f) \xrightarrow{f} r(f)$.
- $i : C_0 \to C_1$ is a set map which associates to an object $A \in C_0$, the identity morphism 1_A.
- $\mu : C_{1,1} \to C_1$ is a set map which provides composition of morphisms as follows:
  - $C_{1,1} = \{(g, f) : g, f \in C_1 \text{ with } l(g) = r(f)\}$ is the collection of pairs of "composable" morphisms.
  - $\mu(g, f)$ is thought of as the composition $g \circ f$ of g and f.
- We have $l \circ i = 1_{C_0} = r \circ i$ which says that the identity morphism is from an object to itself.
- We have $\mu(f, i(l(f)) = \mu(i(r(f)), f) = f$ which says that identity morphisms are left and right identities with respect to composition.
- We have $l(\mu(g, f)) = l(f)$ and $r(\mu(g, f)) = r(g)$ which says that composition produces an arrows from the left of the second part to the right of the first part.

$$l(f) \xrightarrow{f} r(f) = l(g) \xrightarrow{g} r(g), \qquad l(f) \xrightarrow{\mu(g,f)} r(g)$$

- If $l(g) = r(f)$ and $l(h) = r(g)$, then we have $\mu(h, \mu(g, f)) = \mu(\mu(h, g), f)$ which is the associative law for composition.

Note that this definition is similar to other set-theoretic definitions like those of groups, rings, topological spaces, and so on.

Monoids

We have already seen examples of small categories, like the category associated with a set or a group.

Another interesting example is that of a small category with *one* object. Since $C_0 = \{\cdot\}$ is a singleton set, we see easily that the above conditions are often easily satisfied. Thus, we are reduced to a smaller list of conditions as follows.

- $M = C_1$ is a set which has a distinguished element $1_M = i(\cdot)$.
- $\mu : M \times M \to M$ is a set map.
- $\mu(1_M, m) = \mu(m, 1_m) = m$ for every element $m \in M$.
- $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$ for elements a, b, c in M.

A set with such a structure is called a *monoid*. Note the similarity with the notion of a group *except* for the requirement every element has an inverse.

Groupoids

A small category in which every morphism is an isomorphism is called a *groupoid*. Note that this means that we have (C_0, C_1, r, l, i, μ) as above as well as $\iota : C_1 \to C_1$ which represents the inverse and satisfies:

- $\iota \circ \iota = 1_{C_1}$
- $l \circ \iota = r$ and $r \circ \iota = l$
- $\mu(f, \iota(f)) = i(r(f))$ and $\mu(\iota(f), f)) = i(l(f))$

This generalises the category associated to a group by having a group $\mathcal{C}(A, A)$ associated with each object A. One checks that a morphism $f : A \to B$ gives an isomorphism between the groups $\mathcal{C}(A, A)$ and $\mathcal{C}(B, B)$. Note that different morphisms could give different isomorphisms so we *can* only identify these groups upto isomorphism. In particular, a morphism $f : A \to A$ gives the isomorphsim of $\mathcal{C}(A, A)$ to itself given by inner conjugation; as we know from group theory, when this is a non-abelian group, this need not be the identity map.

Functors between Small Categories

A functor F from a small category $\mathcal{C} = (C_0, C_1, r, l, i, \mu)$ to a small category $\mathcal{D} = (D_0, C_1, r, l, i, \mu)$ can be defined as follows:

- Set maps $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ which is the result of the functor being applied to objects and morphisms.
- The condition $i \circ F_0 = F_1 \circ i$ which says that the image of the identity morphism is the identity morphism.
- The conditions $l \circ F_1 = F_0 \circ l$ and $r \circ F_1 = F_0 \circ r$ which say that the left and right of the image morphism are the images of the left and right of the morphisms respectively.
- Given that $l(g) = r(f)$, the condition $F_1(\mu(g, f)) = \mu(F_1(g), F_1(f))$ which says that functor on morphisms preserves composition.

It is clear that functors can be composed and that this composition is associative and has left and right identities. In other words, we can make a category out of small categories!

Natural Transformations

Note that the collection of functors between two small categories $\mathcal{C}$ and $\mathcal{D}$ form a set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. In fact, this is a subset of those elements $F = (F_0, F_1)$ of $\mathrm{Map}(C_0, D_0) \times \mathrm{Map}(C_1, D_1)$ that satisfy the above conditions.

A natural transformation from $F = (F_0, F_1)$ to $G = (G_0, G_1)$ consists of:

- A morphism $\eta : C_0 \to D_1$.
- For an object $A \in C_0$, the conditions $l(\eta(A)) = F_0(A)$ and $r(\eta(A)) = G_0(A)$ which say that $\eta(A)$ is a morphism in $\mathcal{D}$ from $F_0(A)$ to $G_0(A)$.
- For a morphism $f \in C_1$, we let $A = l(f)$ and $B = r(f)$. We have $\mu(G_1(f), \eta(l(f))) = \mu(\eta(r(f)), F_1(f))$ which says that the following diagram commutes

$$\begin{array}{ccc} F_0(A) & \xrightarrow{\eta(A)} & G_0(A) \\ \Big\downarrow{\scriptstyle F_1(f)} & & \Big\downarrow{\scriptstyle G_1(f)} \\ F_0(B) & \xrightarrow{\eta(B)} & G_0(B) \end{array}$$

In other words, natural transformations $\mathrm{Nat}(F, G)$ can be seen as a subset of $\mathrm{Map}(C_0, D_1)$ consisting of those maps η that satisfy the above conditions. We denote the natural transformation by the same symbol η, which is a bit of abuse of notation!

Given $\eta : C_0 \to D_1$ in $\mathrm{Nat}(F, G)$ and $\tau : C_0 \to D_1$ in $\mathrm{Nat}(G, H)$, we define the composite natural transformation $\sigma = \mu(\tau, \eta) : C_0 \to D_1$ which maps A in C_0 to the element $\sigma(A) = \mu(\tau(A), \eta(A))$ in D_1. Note that $l(\tau(A)) = G_0(A) = r(\eta(A))$ so that this composition makes sense. One then checks that σ is an element of $\mathrm{Nat}(F, H)$ as follows:

- For an object $A \in C_0$, we note that

$$l(\sigma(A)) = l(\mu(\tau(A), \eta(A))) = l(\eta(A) = F_0(A)$$

  and

$$r(\sigma(A)) = r(\mu(\tau(A), \eta(A))) = r(\tau(A) = H_0(A)$$

- For a morphism $f \in C_1$, let $A = l(f)$ and $B = r(f)$. We have the commutative diagram

$$\begin{array}{ccccc} & & \overset{\tau(A)}{\frown} & & \\ F_0(A) & \xrightarrow{\eta(A)} & G_0(A) & \xrightarrow{\tau(A)} & H_0(A) \\ \Big\downarrow{\scriptstyle F_1(f)} & & \Big\downarrow{\scriptstyle G_1(f)} & & \Big\downarrow{\scriptstyle H_1(f)} \\ F_0(B) & \xrightarrow{\eta(B)} & G_0(B) & \xrightarrow{\tau(B)} & H_0(B) \\ & & \underset{\tau(B)}{\smile} & & \end{array}$$

We call $i \circ F_0 : C_0 \to D_1$ the identity natural transformation from F to itself as it associates to A, the identity map $1_{F_0(A)} : F_0(A) \to F_0(A)$.

We then check rather easily that this acts as identity for the composition of natural transformations and that the composition of natural transformations is associative.

In other words, the set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the collection of objects of a *category* $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ where natural transformations $\mathrm{Nat}(F, G)$ play the role of morphisms from F to G.

This can appear to be a little complicated to begin with, but it is worth checking all the algebraic identities involved!