Categories, Functors and Natural Transformations

(Loosely based on "Basic Category Theory" by Tom Leinster.)

Categories

A category $\mathcal{C}$ has two species: objects (usually denoted by capital letters) and morphisms (usually denoted by lower case letters). A morphism is a labelled arrow from one object to another and in this sense we can think of a category as a special kind of directed labelled graph. Ojects and morphisms satisfy the following properties:

- Given an object A, there is a morphism $1_A : A \to A$
- Given morphisms $f : A \to B$ and $g : B \to C$, we have a *composite* morphism $gf : A \to C$. (Sometimes we also denote this as $g \circ f$ to make the composition explicit.)
- The composition of morphisms is associative. Given morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have the identiy of the composite morphisms $(hg)f = h(gf) : A \to D$.
- The morphisms 1_A and 1_B are right and left identities for morphisms $f : A \to B$. In other words, we have $1_B \circ f = f = f \circ 1_A : A \to B$

Sets, Groups, Topological spaces

The category **Set** has objects as sets, morphisms as set maps and composition as composition of set maps. It is clear that this satisfies the properties given above.

The category **Grp** has objects as groups, morphisms as group homomorphisms. It is clear that this satisfies the properties given above.

The category **Top** has objects as topological spaces and morphisms as continuous maps. It is clear that this satisfies the properties given above.

We can similarly talk about the category **Ring** of rings (with identity), the category **Rng** of rings without identity and so on.

Constructions from Mathematical Objects

Given a set S we define a category S whose objects are elements of S and the only morphisms are the identity morphisms.

Given a group G, we can construct a category (which we can also denote as G!) which has a single object $\bullet$ and morphisms from this object to itself are elements of the group.

Given a poset P (recall that this means that P is a set with a partial order $\leq$), we define a category P whose objects are elements of P and there is a unique arrow (morphism) from a to b if $a \leq b$.

In particular, for a topological space X we can consider the category associated with the poset $O(X)$ of open sets in X.

Special Categories

We can consider the category $\mathbf{1}$ which has a single object $\bullet$ which has only the identity morphism. This is a special case of the constructions above for the singleton set, the singleton group, the singleton poset and the empty topological space!

We can consider the category $\mathbf{1}_{\rightarrow}$ which has a unique non-identity morphism between two distinct objects.

Given a category $\mathcal{C}$, we have the category $\mathcal{C}^{\text{opp}}$ whose objects are the objects of $\mathcal{C}$, and morphisms are also those of $\mathcal{C}$ but with direction of arrow reversed! We can define composition $\diamond$ by writing $f \diamond g$ for gf. Note that $\mathbf{1}^{\text{opp}} = \mathbf{1}$.

Functors

A functor F from the category $\mathcal{C}$ to the category $\mathcal{D}$ assigns to each object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$, and to each morphism $f : A \to B$ of $\mathcal{C}$ a morphism $F(f) : F(A) \to F(B)$ so that the following properties are satisfied:

- Given morphisms $f : A \to B$ and $g : B \to C$ of $\mathcal{C}$, we have $F(gf) = F(g)F(f) : F(A) \to F(C)$.
- We have $F(1_A) = 1_{F(A)} : F(A) \to F(A)$.

Forgetful functors

Since maps of groups or topological spaces are, in particular, set maps of the underlying sets, we obtain functors $\mathbf{Grp} \to \mathbf{Set}$ (respectively $\mathbf{Top} \to \mathbf{Set}$) that take each group (respectively topological space) to the underlying set and each homomorphism (respectively continuous map) to the underlying map of sets.

We can do similar things with **Ring**, **Rng** and so on.

Constructions from Mathematical Objects

Given sets S and T and a set map $f : S \to T$ we can think of this as a map of the associated categories with objects as elements of the sets and only identity morphisms. Note that conversely, a functor between these categories is *determined* by a set map.

Similarly, we see that if $f : G \to H$ is a homomorphism of groups and we think of G and H as categories with single objects and morphisms as elements, then this gives a functor between these categories. Conversely, a functor between these categories is *precisely* a homomorphism of groups.

Again, if P and Q are posets thought of as categories, then order presering morphisms from P to Q are precisely functors from the category associated with P to the category associated with Q.

The case for the category $O(X)$ of a topological space is more subtle in two ways. First of all, a continuous maps $f : X \to Y$ gives an order preserving map $f^{-1} : O(Y) \to O(X)$ on the collection of opensets; the order is *reversed*. Secondly, an order preserving map $g : O(Y) \to O(X)$ *need not* be of the form $g = f^{-1}$ for some continuous map $f : X \to Y$.

Other functors

A functor from the category $\mathbf{1}$ to a category $\mathcal{C}$ is determined by a choice of an object A in $\mathcal{C}$.

A functor from the category $\mathbf{1}_{\to}$ to a category $\mathcal{C}$ is determined by a choice of an morphism $f : A \to B$ in $\mathcal{C}$. (Note that A and B *need not* be distinct objects in $\mathcal{C}$.)

A functor from the category $\mathcal{C}^{\mathrm{opp}}$ to a category $\mathcal{D}$ is similar to a functor $\mathcal{C}$ to $\mathcal{D}$ *except* that arrows (morphisms) are reversed. In other words, this associates to an object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$, and it associates to a morphism $f : A \to B$ of $\mathcal{C}$ a morphism $F(f) : F(B) \to F(A)$ of $\mathcal{D}$. Such an F is often called a *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$. Note that it is *not* a particular type of functor from $\mathcal{C}$ to $\mathcal{D}$.

G-actions

A functor from the category G associated with a group to a category $\mathcal{C}$ gives:

- An object A of $\mathcal{C}$ which is associated to the single object of G.
- A morphism $\rho(g) : A \to A$ associated to an element g of G.

These need to satisfy the following conditions:

- The identity element of G maps to the identity morphism $1_A : A \to A$
- The composition $\rho(g) \circ \rho(h)$ should be associated to the element $g \cdot h$ of G, where the latter is the product of the elements g and h.

When $\mathcal{C}$ is taken to be the category **Set** of sets, this is precisely the definition of a set A with a (left) action of G. Hence, more generally, such a object can be considered to be an object A of $\mathcal{C}$ with a G-action. With this terminology, functors from G to $\mathcal{C}$ correspond to objects with a given G-action.

Functors to Set

We often restrict our attention to categories such that morphisms between two objects form a set. For example, this is true for **Set** and thus also for **Grp** and **Top**. Note that in these categories, the collection of objects is *not* a set.

Given objects A and B in such a category $\mathcal{C}$, we use $\mathcal{C}(A, B)$, or sometimes $\mathrm{Mor}_{\mathcal{C}}(A, B)$, to denote the set of morphisms from A to B. Note that composition is a set map

$$\circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C) \text{ given by } (g, f) \mapsto g \circ f$$

Fixing an object A of $\mathcal{C}$, we get a functor $A_\cdot$ from $\mathcal{C}$ to **Set** that associates, to an object B, the set $A_\cdot(B) = \mathcal{C}(A, B)$, and to a morphism $g : B \to C$, the set map

$$A_\cdot(g) = g \circ \_ : \mathcal{C}(A, B) \to \mathcal{C}(A, C) \text{ given by } f \mapsto g \circ f$$

We also have a *contravariant* functor $A^\cdot$ which associates, to an object B. the set $A^\cdot(B) = \mathcal{C}(B, A)$, and to a morphism $g : B \to C$, the set map

$$A^\cdot(g) = \_ \circ g : \mathcal{C}(C, A) \to \mathcal{C}(B, A) \text{ given by } h \mapsto h \circ g$$

Note that this is a functor from $\mathcal{C}^{\mathrm{opp}}$ to **Set**.

Natural Transformations

It is said that *Category Theory* was invented by Eilenberg and Maclane in order to formalise the intuitive notion of "natural" transformations as detailed below.

Given functors F and G from $\mathcal{C}$ to $\mathcal{D}$, a natural transformation $\eta : F \to G$ gives, for each object A of $\mathcal{C}$, a morphism $\eta_A : F(A) \to G(A)$, and for each morphism $f : A \to B$, a commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

Recall that a diagram is said to be commutative if any path following the arrows from one vertex to another gives the *same* morphism. In this case, this is another way of saying that $\eta_B \circ F(f) = G(f) \circ \eta_A$.

Morphisms of $\mathcal{C}$ as natural transformations

As seen above, an object A of $\mathcal{C}$ can be seen as a functor **1** to $\mathcal{C}$ that assigns, to the unique object of **1**, the object A of $\mathcal{C}$ and to the unique (identity) map of **1**, the identity morphism 1_A.

A morphism $f : A \to B$ can then be seen as a natural transformation from the functor associated to A to the functor associated to B. This natural transformation assigns, to the unique object of **1**, the morphism f. The commutativity of the above diagram is then the standard identity relation $f \circ 1_A = 1_B \circ f$.

Conversely, a natural transformation between these functors gives a morphism $f : A \to B$ and the functor is *precisely* the one associated with this morphism.

G-equivariant maps as natural transformations

As seen above, a set S with a left action of a group G can be viewed as a functor from the category associated with G to the category **Set** that associates to the unique object of G, the set S, and to an element g in G, the left-action $\rho(g) : S \to S$ considered as a (bijective) map of sets.

Suppose that another set T has the action given by $\tau(g) : T \to T$ for g in G.

A natural transformation between such functors associated with S and T is, in particular, a set map $f : S \to T$. In *addition*, the commutativity of the above digram means that $\tau(g) \circ f = f \circ \rho(g)$. This is *precisely* the condition that the map $f : S \to T$ is G-equivariant. Writing the action as left multiplication without ρ and τ, this equation becomes $f(g \cdot s) = g \cdot f(s)$.

Natural Transformations for the "dot" Functors to Sets

Given a category $\mathcal{C}$ where the morphisms between objects form sets, we introduced the functor $A_{\cdot}$ from $\mathcal{C}$ to **Set** and the functor $A^{\cdot}$ from $\mathcal{C}^{\text{opp}}$ to **Set**.

A natural transformation $A_{\cdot} \to B_{\cdot}$, gives in particular, a set map $A_{\cdot}(A) = \mathcal{C}(A, A) \to \mathcal{C}(B, A) = B_{\cdot}(A)$. The image of the identity element 1_A in $\mathcal{C}(A, A)$ gives a morphism $f : B \to A$ associated to this natural transformation. The above commutative diagram can then be used to show that this set map is *precisely* $h \mapsto h \circ f$.

Conversely, given a morphism $f : B \to A$, we can give, for each object C, the set map

$$A_{\cdot}(C) = \mathcal{C}(A, C) \to B_{\cdot}(C) = \mathcal{C}(B, C) \text{ given by } h \mapsto h \circ f$$

The associative law for composition can be used to show that this yields a commutative diagram as required. In fact, this construction *identifies* natural transformations $A_{\cdot} \to B_{\cdot}$ with morphisms $B \to A$.

Similarly, one can show that a natural transformation $A^{\cdot} \to B^{\cdot}$ can be identified with a morphism $A \to B$.

The above statements are versions of the Yoneda lemma which we shall explain a little later.