$\mathbb{Z}$-Schemes

MTH437 — Introduction to Schemes

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Recall

In the previous lecture we provided two examples to show how $\mathbb{Z}$-affine schemes can be patched to give $\mathbb{A}^2 \setminus \{(0,0)\}$ and $\mathbb{P}^1$.

We now generalise this to a definition and provide some further examples.

Equivalence Relation on a functor

Given F a functor **CRing** to **Set**, a subfunctor $E \subset F \times F$ is said to be an equivalence relation on the functor F, if $E(R) \subset F(R) \times F(R)$ is an equivalence relation on $F(R)$ for each ring R.

Recall that this means that:

- (Reflexive) For each a in $F(R)$ we have (a, a) in $E(R)$.
- (Symmetric) For (a, b) in $E(R)$ we have (b, a) in $E(R)$.
- (Transitive) For (a, b) and (b, c) in $E(R)$ we have (a, c) in $E(R)$.

We wish to study some examples of such equivalence relations and the notion of quotient by an equivalence relation.

Given a natural transformation $\eta : F \to G$ of functors, define

$$E(R) = \{(a, b) \in F(R) \times F(R) : \eta_R(a) = \eta_R(b) \text{ in } G(R)\}$$

It is clear that $E(R)$ is an equivalence relation on $F(R)$.

We now need to prove that E is a functor. Given a ring homomorphism $f : R \to S$, and (a, b) in $E(R)$:

- $\eta_R(a) = \eta_R(b)$ in $G(R)$, so
- $G(f)(\eta_R(a)) = G(f)(\eta_R(b))$ in $G(S)$, and
- $G(f) \circ \eta_R = \eta_S \circ F(f)$ since $\eta : F \to G$ is a natural transformation
- $\eta_S(F(f)(a)) = \eta_S(F(f)(b))$ in $G(S)$, so
- $(F(f)(a), F(f)(b))$ is in $E(S)$.

Thus, $F(f) \times F(f)$ maps $E(R)$ to $E(S)$. So E is a subfunctor of $F \times F$. Hence, E is an equivalence relation on the functor F.

We want to think of the image of $\eta : F \to G$ as the *quotient* of F by E.

However, the functor $F(R)/E(R)$ is not, in general, a sheaf even when G, F and E are sheaves.

So we want to show how to construct a sheaf functor F/E which *represents* the quotient in a geometric sense.

Example: $\mathbb{Z}$-quasi affine scheme as a union

Given a $\mathbb{Z}$-quasi affine scheme

$$Q = A(x_1, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$$

which is a subscheme of the $\mathbb{Z}$-affine scheme $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q)$.

For $i = 1, \ldots, r$ we define the $\mathbb{Z}$-affine schemes

$$U_i = A(x_1, \ldots, x_p, v_i; f_1, \ldots, f_q, v_i g_i - 1)$$

which are affine open subschemes of X.

We wish to exhibit Q as the "union of U_i in X".

First of all note that $U = \sqcup_i U_i$ is an affine scheme and we have a natural morphism $q : U \to X$ which restricts to the inclusions $U_i \to X$ for each i.

So we can see Q as the "image of q".

As above, we define the *equivalence relation* V on U such that $V(R)$ consists of pairs of points (a, b) in $U(R)$ such that $q(a) = q(b)$.

We will show that $V(R)$ is the R-points of a subscheme V in $U \times U$.

Note that $U \times U = \sqcup_{i=1}^{r} \sqcup_{j=1}^{r} U_i \times U_j$.

For $i, j = 1, \ldots, r$ define the $\mathbb{Z}$-affine schemes

$$U_{i,j} = A(x_1, \ldots, x_p, v_i, v_j; f_1, \ldots, f_q, v_i g_i - 1, v_j g_j - 1)$$

which can be identified with $U_i \cap U_j$ in X.

Now, consider the scheme $V = \sqcup_{i=1}^{r} \sqcup_{j=1}^{r} U_{i,j}$.

Note that $U_{i,i}$ is just U_i, and $U_{i,j} = U_{j,i}$.

We have natural morphisms $\delta_{i,j} : U_{i,j} \to U_i \times U_j$ given by "pairing" the inclusions $U_{i,j} \to U_i$ and $U_{i,j} \to U_j$.

Combining all these, we see that we have a morphism $V \to U \times U$ making this a subscheme.

One sees that for each R, the subset $V(R)$ is the subset of $U(R) \times V(R)$ defined above.

Now $U(R)/V(R)$ is *not* the "correct" subset $Q(R)$ of $X(R)$!

We note that a point of $U(R)$ is a point of $X(R)$ where *at least* one of the functions g_i has a unit as its image in R.

However, a point of $Q(R)$ is a point where $\langle g_1, \ldots, g_r \rangle_R = R$.

The "sheaf" version U/V of the quotient will solve this issue.

Example: Patching affine schemes

We can generalise the above idea to define the notion of schemes via patching as follows.

We wish to patch the following data:

- Given $\mathbb{Z}$-affine schemes U_i for $i = 1, \ldots, k$.
- Given $\mathbb{Z}$-affine schemes $U_{i,j}$ for $1 \leq i < j \leq k$.
- Given inclusions $U_{i,j} \to U_i$ making this an affine open subscheme.
- Given inclusions $U_{i,j} \to U_j$ making this an affine open subscheme.

…with one additional condition below.

For convenience, let us define $U_{i,i}$ as U_i and $U_{j,i} = U_{i,j}$ for $1 \leq i < j \leq k$.

As above, we define $U = \sqcup_i U_i$ and $V = \sqcup_i \sqcup_j U_{i,j}$.

We have the inclusions $\delta_{i,j} : U_{i,j} \to U_i \times U_j$ which we combine as before to give V as a subfunctor of $U \times U$.

The additional condition we require is that $V(R)$ is an equivalence relation on $U(R) \times U(R)$ for each ring R.

Note that V is already reflexive and symmetric. Hence, the missing condition is the transitivity.

The transitivity condition ensures that $U_{i,j}$, $U_{j,k}$ and $U_{i,k}$ are related so as to correctly patch and obtain the "union" of U_i, U_j and U_k along them.

As before, we would like to construct the sheaf quotient U/V as the geometric representation of the result of patching the U_i's.

The Zariski sheaf quotient

We now "construct" the Zariski sheaf quotient of a functor F by an equivalence relation on it given by a functor E.

For a ring R, a collection

$$(\mathbf{u}, \mathbf{h}) = ((u_1, \ldots, u_k), (h_1, \ldots, h_k))$$

of "descent data" for $(E/F)(R)$ are described as follows:

- We are given elements $u_1, \ldots, u_k$ in R such that $\langle u_1, \ldots, u_k \rangle = R$.
- We have points h_i in $U(R_{u_i})$ for $i = 1, \ldots, k$.
- For each i and j, the pairs (h_i, h_j) in $(U \times U)(R_{u_i u_j})$ lie in $V(R_{u_i u_j})$.

We want to define $(E/F)(R)$ to be the set of descent data *modulo* a certain equivalence relation which we now describe.

Given *another* set of elements $v_1, \ldots, v_m$ in R such that $\langle v_1, \ldots, v_m \rangle = R$, we note that if we define $w_{i,j} = u_i v_j$, then the collection of $w_{i,j}$ *also* generate the unit ideal in R.

Let $g_{i,j}$ be the image of h_i via the set map $U(R_{u_i}) \to U(R_{w_{i,j}})$.

The pairs $(g_{i,j}, g_{r,s})$ are the images of (h_i, h_r) via the set map $V(R_{u_i u_r}) \to V(R_{wi,jw_{r,s}})$.

Thus, we see that $(\mathbf{w}, \mathbf{g})$ defined using $(w_{i,j})$ and $(g_{i,j})$ are also descent data for $(E/F)(R)$.

We say that $(\mathbf{w}, \mathbf{g})$ is a *refinement* of the descent data $(\mathbf{u}, \mathbf{h})$ using the tuple $(\mathbf{v})$.

Given two descent data $(\mathbf{u}, \mathbf{h})$ and $(\mathbf{v}, \mathbf{f})$ we can form:

- the refinement $(\mathbf{w}, \mathbf{g})$ of $(\mathbf{u}, \mathbf{h})$ using the tuple $\mathbf{v}$.
- the refinement $(\mathbf{w}, \mathbf{e})$ of $(\mathbf{v}, \mathbf{f})$ using the tuple $\mathbf{u}$.

Note that $w_{i,j} = u_i v_j$ are the same in both refinements.

We declare the above two sets to be *equivalent* if $(g_{i,j}, e_{i,j})$ lie in $V(R_{w_{i,j}})$.

As mentioned earlier $(E/F)(R)$ is the quotient of the collection of descent data by this equivalence relation.

The above definition is made more complicated by the fact that we need to have a lot of notation for the subscripts! It is very similar to the definition of manifolds using the idea of an atlas.

The construction seems complex at first glance. However, the main thing to note is that there *is* a Zariski sheaf that represents the quotient F/E.

Perhaps the following interpretation of the sheaf condition will clarify the matter.

Interpretation of sheaf condition

Consider a functor **CRing** to **Set** which has the following property.

Zariski Sheaf property of F: Given a commutative ring R and elements $u_1, \ldots, u_k$ which generate the unit ideal. Given $h_i \in F(R_{u_i})$ for $i = 1, \ldots, k$ such that the image of h_i and h_j in $F(R_{u_i u_j})$ are the same, there is a unique point h in $F(R)$ so that h_i is its image in $F(R_{u_i})$

Suppose $R = \mathcal{O}(X)$ where $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q)$ is a $\mathbb{Z}$-affine scheme.

Then $R_{u_i} = \mathcal{O}(U_i)$ where $U_i = A(x_1, \ldots, x_p, v_i; f_1, \ldots, f_q, v_i u_i - 1)$ is the $\mathbb{Z}$-affine subscheme of X where u_i is non-zero (i.e. a unit). In other words, this is an affine open subscheme of X.

Similarly $R_{u_i u_j} = \mathcal{O}(U_{i,j})$ where $U_{i,j} = U_i \cap U_j$ in X.

By the Yoneda lemma, h_i can be seen as morphisms (natural transformations) $U_i \to F$.

The condition that h_i and h_j have the same image in $F(R_{u_i u_j})$ can be written as

$$h_{i|_{U_i \cap U_j}} = h_{j|_{U_i \cap U_j}}$$

We have seen that the condition that $\langle u_1, \ldots, u_k \rangle = R$ can be interpreted to mean that X is "covered by the union of U_i".

The Zariski sheaf condition can thus be interpreted as saying:

> *Given a covering of an affine scheme X by affine open subschemes U_i for $i = 1, \ldots, k$ and morphisms $h_i : U_i \to F$ which satisfy*
>
> $$h_{i|_{U_i \cap U_j}} = h_{j|_{U_i \cap U_j}}$$
>
> *there is a unique morphism $h : X \to F$ which restricts to h_i on U_i.*

Conclusion

There are two closely related ways in which patching plays a role.

1. A $\mathbb{Z}$-scheme is defined by patching together $\mathbb{Z}$-affine schemes.
2. The sheaf condition for a functor F says that a morphism (natural transformation) to F from a $\mathbb{Z}$-affine scheme is obtained by patching together morphisms on an open cover.

Since the categorical point of view says that geometry and morphisms determine each other, we see that these two roles are essentially the same!