Beyond Affine Schemes

MTH437 — Introduction to Schemes

Kapil Hari Paranjape

IISER Mohali

30th September 2021

Recall

We showed that a scheme X can be identified with a functor **CRing** to **Set** which we denoted as $X_{\cdot}$.

A morphism $f : X \to Y$ can be identified with a natural transformation $\tilde{f} : X_{\cdot} \to Y_{\cdot}$.

We showed that such functors satisfy:

Sheaf property of Schemes: Given a commutative ring R and elements $u_1, \ldots, u_k$ which generate the unit ideal. Given R_{u_i}-points h_i of X for $i = 1, \ldots, k$ such that the image of h_i and h_j in $X(R_{u_i u_j})$ are the same, there is a unique point h in $X(R)$ so that h_i is its image in $X(R_{u_i})$

This will lead us to define (general) schemes as functors **CRing** to **Set** which satisfy this property (... and some additional conditions).

Two Questions

Q1: Why do we want to extend the category of $\mathbb{Z}$-affine schemes?

Q2: Why is the above solution the "right" one?

We want to motivate/justify the (currently) chosen answer to these questions!

Most basic results about schemes will fall into place once we know the answers somewhat.

Geometric meaning of $\mathbb{Z}$-affine schemes

We think of $\mathbb{A}^p = A(x_1, \ldots, x_p;)$ as the $\mathbb{Z}$-Affine scheme which represents p-dimensional affine space.

(Note that $\mathbb{A}^p(R) = R^p$ for every ring R.)

In this sense a scheme $X = A(x_1, \ldots, x_p; f_1, \ldots, f_p)$ represents the locus of *simultaneous* zeroes of $f_1, \ldots, f_p$.

However, these are not the only types of loci that we want to study.

Loci defined by equalities and identities are "closed".

We also need to look at "open" loci.

- We can think of these as being defined by (strict) inequalities.
- We can also think of these as complements of closed loci.

Example

In geometry, we would like to study various naturally arising parameter spaces.

For example, we would like to study the space of all ordered pairs of *distinct* points in the affine plane.

We can think of $\mathbb{A}^4 = A(x, y, z, w;)$ as the space of all pairs $((x, y), (z, w))$ of points in the plane.

The affine scheme $\Delta = A(x, y, z, w; x - z, y - w)$ is the space of all those pairs which are two copies of the *same* point.

The space of pairs of distinct points is the *complement* $\mathbb{A}^4 \setminus \Delta$.

Algebraic Inequalities

When we hear of inequalities, we think of things like $x > 0$.

However, this does not make sense for *all* rings.

- For an order to be algebraically useful, we needi (for addition) $a > b \implies a + c > b + c$.
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ have natural orders.
- $\mathbb{C}$, $\mathbb{F}_p$ have no *possible* orders.

Since we want to work with all rings, we *could* think of conditions like $x \neq 0$.

The problem with this is that under a ring homomorphism $R \to S$ a non-zero element can go to 0.

So a “solution” of the inequality $x \neq 0$ in R may not have an image in S.

We thus replace $x \neq 0$ with the requirement that x is a unit. Equivalently, we put an *equation* of the form $xt - 1 = 0$ for a new variable t.

What do we do if we want the inequality $(x, y) \neq (0, 0)$?

In analogy with the above, we *could* add variables u and v and add the equation $xu + yv = 1$.

Note that this says that $\langle x, y\rangle$ is the unit ideal.

However, unlike $xu - 1 = 0$ which has a *unique* solution for u given a unit x, there are many (u, v) pairs that are associated with the *same* pair (x, y).

For example, $x(u + ay) + y(v - ax) = 1$ shows that $(u + ay, v - ax)$ is another solution for any value of a.

However, we shall see that the "functor of points" approach gives a resolution of this problem.

Quasi-affine $\mathbb{Z}$-scheme

A quasi-affine $\mathbb{Z}$-scheme is denoted $A(x_1, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$.

We think of this as the locus of points in $\mathbb{A}^p$ which satisfy the equations $f_i = 0$ for $i - 1, \ldots, q$ and the "inequality" $\langle g_1, \ldots, g_r \rangle = \langle 1 \rangle$.

To the quasi-affine scheme $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$ we associate a functor of points $X_.$ from **CRing** to **Set**.

The set $X_.(R)$ consists of tuples $(a_1, \ldots, a_p)$ such that there *exist* elements $b_1, \ldots, b_r$ in R for which $x_i \mapsto a_i$ and $y_j \mapsto b_i$ gives a ring homomorphism:

$$\frac{\mathbb{Z}[x_1, \ldots, x_p, y_1, \ldots, y_r]}{\left\langle f_1, \ldots, f_p, 1 - \sum_{j=1}^{s} y_j g_j(x_1, \ldots, x_p) \right\rangle} \to R$$

Note that another choice of $c_1, \ldots, c_r$ of images for $y_1, \ldots, y_r$ gives the *same* element of $X_.(R)$.

Given a ring homomorphism $f : R \to S$, we have a map which sends $(a_1, \ldots, a_p)$ to $(f(a_1), \ldots, f(a_p))$ as before.

We only need to note that $f(b_1), \ldots, f(b_r)$ give *one possible choice* for the required images of $y_1, \ldots, y_r$ in S.

Another way to say this is that if $Y = A(x_1, \ldots, x_p; f_1, \ldots, f_q)$ is the associated $\mathbb{Z}$-affine scheme, then

$$X_.(R) = \{\mathbf{a} \in Y_.(R) \mid |\langle g_1(\mathbf{a}), \ldots, g_r(\mathbf{a})\rangle_R = R\}$$

Set-theoretic constructions

Now that we are studying functors **CRing** to **Set** we can ask if some usual set-theoretic constructions have a natural meaning.

Given F and G are two such functors, we can define $F \times G$ in a natural way:

- For a ring R, we have $(F \times G)(R) = F(R) \times G(R)$.
- For a ring homomorphism $f : R \to S$, we have $(F \times G)(f) = F(f) \times G(f)$ where

$$F(f) \times G(f)(x, y) = (F(f)(x), G(f)(y)) \text{ where } (x, y) \in X(R) \times Y(R)$$

When $F = X_{\cdot}$ and $G = Y_{\cdot}$ are $\mathbb{Z}$-affine schemes, what is $F \times G$?

Suppose $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q)$ and $Y = A(y_1, \ldots, y_r; g_1, \ldots, g_s)$.

Consider the $\mathbb{Z}$-affine scheme
$Z = A(x_1, \ldots, x_p, y_1, \ldots, y_r; f_1, \ldots, f_p, g_1, \ldots, g_s)$.

We check easily that $Z_\cdot$ and be naturally identified with $X_\cdot \times Y_\cdot$.

Equivalence relations and quotients

One way to construct a new set from an existing set is by taking quotient by an equivalence relation.

An equivalence relation E on a set X is a subset of $X \times X$ such that:

1. For $x \in X$, the pair (x, x) lies in E. This is called *reflexivity*.
2. If the pair (x, y) is in E, then so is (y, x). This is called *symmetry*.
3. If the pairs (x, y) and (y, z) are in E, then so is (x, z). This is called *transitivity*.

Given $x \in X$, the set $E_x = \{y : (x, y) \in E\}$ is called the equivalence class of x.

This gives a map $X \to P(X)$ given by $x \mapsto E_x$. The image of this map is denoted X/E and is called the quotient of X by E.

Note that $f : X \to X/E$ is a surjective (onto) map of sets.

Conversely, given a surjective map $f : X \to Y$, we can define

$$E = \{(x, x') : f(x) = f(x')\}$$

and check that this is an equivalence relation.

Moreover, we check that there is a natural bijection $X/E \to Y$.

Example

Given $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$ a $\mathbb{Z}$-quasi-affine scheme we can consider the $\mathbb{Z}$-affine scheme

$$\tilde{X} = A(x_1, \ldots, x_p, y_1, \ldots, y_r; f_1, \ldots, f_q, 1 - \sum_j y_j g_j)$$

We clearly have an onto map $\tilde{X}(R) \to X(R)$ by "forgetting" the assignments of values in R to the variables $y_1, \ldots, y_r$. This is how we *defined* $X(R)$.

Now, $\tilde{X}(R)$ carries an equivalence relation $E(R)$ which consists of pairs of points in $\tilde{X}(R)$ that correspond to the same point in $X(R)$.

As seen above, this means that $X(R)$ can be identified with $\tilde{X}(R)/E(R)$.

Note that if

$$E = A\left(\mathbf{x}, \mathbf{y}, \mathbf{z}; f_1, \ldots, f_q, 1 - \sum_j y_j g_j(\mathbf{x}), 1 - \sum_j z_j g_j(\mathbf{x})\right)$$

then $E(R)$ is exactly as above.

So $E(R)$ can *also* be seen as the R-valued points of an $\mathbb{Z}$-affine scheme.

In other words, we have an equivalence relation on $\tilde{X}_\cdot$ which is *represented* by a functor $E_\cdot$.

This suggests that we should consider *expanding* our notion of schemes to include "quotients" of $\mathbb{Z}$-affine schemes by equivalence relations which are also $\mathbb{Z}$-affine schemes.

We will discuss this in the next lecture.