Rishov

Why do we need R to be commutative to make sense of evaluating f at a specific p-tuple?

Anunoy

What I feel like about the commutativity of R is that since the underlying polynomials are from a polynomial ring over the set of integers and we want to see the elements of X(R) as elements of the kernel of some evaluation maps, these maps must be defined as a ring homomorphism, requiring the ring R to be commutative.

$(a,b) \in R$ ab need not be equal to ba

$x^5y^7 \ominus x^4y^6xy$ $a^5b^7 \overset{?}{=} a^4b^6ab$

"Non-commutative polynomial ring"

x, y, z ~~sequence~~ of "words" in x, y, z

$\mathbb{Z} \oplus \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}xy \oplus \mathbb{Z}x^2 \oplus \mathbb{Z}yz \oplus \mathbb{Z}yx \ldots \oplus \ldots$

$\mathbb{Z}\{x,y,z\} \to R$ "Non-commutative"

$x \to a$

$y \to b$

$z \to c$

$\mathbb{Z}[x,y,z] \to R$ even if R is not commutative

$R = M_{5\times 5}(\mathbb{Z})$

$x \to a$

$y \to b$

$z \to c$ } - have to commute

$\mathcal{O}(X) \to R$

$\dfrac{\mathbb{Z}[x_1,..,x_p]}{\langle f_1,..,f_q\rangle} \to R$

$x_i \to a_i$

$a_1,..,a_p$ must commute with each other.

To evaluate a polynomial at $a_1,..,a_p$ → have to commute!

$$\mathbb{A}^{p}_{\mathbb{Z}} = \mathbb{A}(x_1, \dots, x_p ;)$$

$$\mathbb{A}^{k^2}_{\mathbb{Z}} = \mathbb{A}\left((x_{ij})^{k\ \ k}_{i=1, j=1} ;\right)$$

$$\cancel{x_{ij}\,x_{j1}} \left(\sum_{j=1}^{k} x_{ij} x_{kj}\right) = (i,k)\text{th entry of } (XX^t)$$
$$= \delta_{ik}$$

$$O_{\mathbb{Z}}(k) = \mathbb{A}\left((x_{ij})^{k\ \ k}_{i=1, j=1} ; \left(\sum_{j=1}^{k} x_{ij} x_{rj} - \delta_{ir}\right)^{k}_{\substack{i=1\\ r=1}}\right)$$

Orthogonal group as an affine scheme

$$SL_{\mathbb{Z}}(k) = \mathbb{A}\left(((x_{ij})^{k\ \ k}_{i=1, j=1} ; \det((x_{ij})) - 1\right)$$

$$O_{\mathbb{Z}}(k) \times O_{\mathbb{Z}}(k) \xrightarrow{\mu} O_{\mathbb{Z}}(k)$$
$$((x_{ij}), (y_{ab})) \longrightarrow \left(\left(\sum_j x_{ij} y_{jb}\right)_{i,b}\right)$$

$$O_{\mathbb{Z}}(k) \xrightarrow{t} O_{\mathbb{Z}}(k)$$
$$((x_{ij})) \longrightarrow ((x_{ji}))$$

$$SL_{\mathbb{Z}}(k) \xrightarrow{t} SL_{\mathbb{Z}}(k)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Matrix of co-minors

$$(x_{ij}) \longmapsto (f_{ij})$$

$f_{ij} = (-1)^{i+j} \det\big(X \mid$ leave out ith row & jth col $\big)$

$$GL_{\mathbb{Z}}(1)(R) = \text{units in } R$$

$$\|$$

$\mathbb{A}(x, y ; xy - 1)$ "hyperbola"

$$\mu\big(((x,y), (x', y'))\big) = (xx', yy')$$

$$\iota((x,y)) = (y, x)$$

$$\frac{\mathbb{Z}[x,y]}{(xy-1)} \longrightarrow R \qquad \begin{array}{l} x \to \text{unit } u \\ y \to u^{-1} \end{array}$$

$SL_{\mathbb{Z}}(2) = A(a,b,c,d;\ ad-bc-1)$

$O_{\mathbb{Z}}(2) = A\left(a,b,c,d;\ a^2+b^2-1,\ c^2+d^2-1,\ ac+bd\right)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$B_{\mathbb{Z}}(2) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} ;\ ad-1 \right\}$

$\cap$

$SL_{\mathbb{Z}}(2)$

$$\boxed{\begin{array}{c} \text{Hyperbola} \times \mathbb{A}^1 \\ \shortparallel \\ \mathbb{G}_m = GL_{\mathbb{Z}}(1) \end{array}}$$

as a space

but multiplication is extra

$\mathbb{Z}$-affine Algebraic groups

$\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \xrightarrow{\alpha} \mathbb{A}^1_{\mathbb{Z}}$

$(x,y) \xmapsto{\alpha} (x+y)$

$(x,y) \xmapsto{\mu} (xy)$

$\mathbb{A}^1_{\mathbb{Z}} \xrightarrow{\iota} \mathbb{A}^1_{\mathbb{Z}}$

$x \longmapsto -x$

Morphism.

Functors CRing $\to$ Sets

CRing $\to$ Groups

CRing $\to$ Ring

$G = O_{\mathbb{Z}}(k)$

$G(R)$ is a group.

for any commutative ring R

$O_{\mathbb{Z}}(k)(\mathbb{Z}/_{120}\mathbb{Z})$

finite group.

$0 \to 0$

$1 \to 1$

$\mathbb{Z} \to R$

$SL_{\mathbb{Z}}(2)(\mathbb{Z}/n\mathbb{Z})$

$\mathbb{A}^1_{\mathbb{Z}}(R) = \mathrm{Hom}(\mathbb{Z}[x], R) = R$

$\mathcal{O}_{\mathbb{A}^1_{\mathbb{Z}}} = \mathbb{Z}[x]$

$\mathbb{A}^1_{\mathbb{Z}} = A(x;)$

Topology on R:

$R = \mathbb{R}, \mathbb{C}, \mathbb{Q}_p,$

$\mathbb{R}^n$ (induced) product topology

Polynomials are continuous functions.)

"Topological Ring / Field"

$\mathbb{R} \times \mathbb{R} \xrightarrow[\alpha, \mu]{} \mathbb{R}$ $\mathbb{R} \xrightarrow{i} \mathbb{R}$

are continuous.

$\Leftarrow$

Polynomials are continuous function.

$X(\mathbb{R}), \; X(\mathbb{C}), \; X(\mathbb{Q}_p)$

$X(R_\tau)$ R_τ top. ring $\mathbb{A}$ = ring of adeles.

$X(\mathbb{A})$ top space

Topological spaces!

"Strong" Topologies → "more cont. functs."

CRing

$\mathrm{TopCRing} \xrightarrow{X} \mathrm{Top}$

$R \to \mathbb{R}^n$ ~~$\mathbb{R}$~~

~~Measure space~~

~~$\mathbb{R}^n$ — "~~

Patching

$\mathbb{Z}_{\ell}$

Prakam

X

Can we view Rings as functor.

$CRing \rightsquigarrow Set$

$R \rightsquigarrow X(R) = Hom(\underline{\underline{U(X)}}, R)$

S is a ring

$R \rightsquigarrow Hom(S,R) = S^{\cdot}(R)$

$$\begin{tikzcd} R_1 \arrow[r] & R_2 \end{tikzcd} \rightsquigarrow \begin{tikzcd} Hom(S,R_1) = S^{\cdot}(R_1) \arrow[d] \\ Hom(S,R_2) = S^{\cdot}(R_2) \end{tikzcd}$$

$S \to R_1$

Any ring gives a functor $\underset{S}{\ }$ $CRing \rightsquigarrow Set$

Satisfies Patching.

$$S^{\cdot}(R) \xrightarrow[(f_i)]{} S^{\cdot}(R_{u_i}) \longrightarrow S^{\cdot}(R_{u_i u_j})$$

$$(h_i) \longmapsto (g_{i,j} \circ h_i - g_{j,i}(h_j))$$

$$\begin{tikzcd} h_i : S \arrow[r, "h_i"] \arrow[d, "h_j"'] \arrow[dr, phantom, "\circlearrowleft"] & R_{u_i} \arrow[d] \\ R_{u_j} \arrow[r] & R_{u_i u_j} \end{tikzcd}$$

$\langle u_1, \dots, u_k \rangle = R$

$\sum u_i x_i = 1.$

$\Rightarrow R \quad h \to R$

$S^{\cdot}$ is the scheme "Spec(S)"

is not $\mathbb{Z}$-affine

$S = \mathbb{R}$, $(\mathbb{R}[X,Y]/\langle X^2+Y^2-1\rangle)$

$S = \mathbb{C}$, Spec

Spec($\mathbb{R}$), Spec($\mathbb{C}$)

$CRing \rightsquigarrow Set$

which satisfy patching.

"local" condn:

G = generators and relations

= Multiplication table

= axiomatic def.

Group as a category.

g

Object = Singleton

morphisms = group elements

Group is a category in which there is one object & every morphism is an isomorphism.

Category with only one object.

$\mathcal{C}$ $\quad f$ $\quad f \circ g$ $\quad 1_X$

→ Monoid

so monoid

Category in which every morphism is an isomorphism.

Groupoid

$a \xrightarrow{f} b$, $\quad b \xrightarrow{f^{-1}} a$

$\mathrm{Mor}(a,a) = G_a$

$\mathrm{Mor}(b,b) = G_b$

$\mathrm{Mor}(a,a) \times \mathrm{Mor}(a,b) \times M(b,b)$

Many definitions of the same mathematical object

$\mathbb{R}$ — Dedekind cuts

— Cauchy sequences (order)

metric

$\mathbb{Q} \to \mathbb{Q}_p$.

Schemes — "locally ring spaces"

— Functors.

in Analysis most spaces are closed subspaces of ℓ^∞

↳ instead define in general topological spaces

$\boxed{\mathbb{R}^4}$

Subset G of permutation group S_n

Closed under multiplication

& contains 1