

Rishov

Why do we need R to be commutative to make sense of evaluating f at a specific p-tuple?

Anunoy

What I feel like about the commutativity of R is that since the underlying polynomials are from a polynomial ring over the set of integers and we want to see the elements of  $X(R)$  as elements of the kernel of some evaluation maps, these maps must be defined as a ring homomorphism, requiring the ring R to be commutative.

$$(a, b) \in R \quad ab \text{ need not be equal to } ba$$

$$x^5 y^7 \neq x^7 y^5 \quad a^5 b^7 \neq a^7 b^5$$

"Non-commutative polynomial ring"

$x, y, z$  sequence of "words" in  $x, y, z$

$$\mathbb{Z} \oplus \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}xy \oplus \mathbb{Z}x^2 \oplus \mathbb{Z}yz \oplus \mathbb{Z}yx$$

$$\oplus \dots$$

$$\mathbb{Z}\langle x, y, z \rangle \rightarrow R$$

$$\begin{matrix} x \rightarrow a \\ y \rightarrow b \\ z \rightarrow c \end{matrix}$$

"Non-commutative"

$$\mathbb{Z}\langle x, y, z \rangle \rightarrow R \quad \text{even if } R \text{ is not commutative}$$

$$\begin{matrix} x \rightarrow a \\ y \rightarrow b \\ z \rightarrow c \end{matrix} \quad R = M_{5 \times 5}(\mathbb{Z})$$

- have to commute

$$\mathbb{Z}\langle x \rangle \rightarrow R$$

$$\mathbb{Z}[x_1, \dots, x_p] \rightarrow R$$

$$\langle f_1, \dots, f_s \rangle$$

$$x_i \rightarrow a_i$$

$a_1, \dots, a_p$  must commute with each other.

To evaluate a polynomial at  $a_1, \dots, a_p$  have to commute!

$$A_{\mathbb{Z}}^p = A(x_1, \dots, x_p;)$$

$$A_{\mathbb{Z}}^{k^2} = A\left((x_{ij})_{i=1, j=1}^k; \right)$$

$$\sum_{j=1}^k x_{ij} x_{kj} = (\delta_{ik}) \text{ (Kronecker delta)}$$

$$O_{\mathbb{Z}}(k) = A\left((x_{ij})_{i=1, j=1}^k; \left(\sum_{j=1}^k x_{ij} x_{kj} - \delta_{ik}\right)_{i=1, k=1}^k\right)$$

Orthogonal group as an affine scheme

$$SL_{\mathbb{Z}}(k) = A\left((x_{ij})_{i=1, j=1}^k; \det((x_{ij})) = 1\right)$$

$$O_{\mathbb{Z}}(k) \times O_{\mathbb{Z}}(k) \xrightarrow{\mu} O_{\mathbb{Z}}(k)$$

$$\left((x_{ij}), (y_{ab})\right) \rightarrow \left(\sum_j x_{ij} y_{jb}\right)_{i, b}$$

$$O_{\mathbb{Z}}(k) \xrightarrow{\mu} O_{\mathbb{Z}}(k)$$

$$\left((x_{ij})\right) \rightarrow \left((x_{ji})\right)$$

$$SL_{\mathbb{Z}}(k) \xrightarrow{\mu} \otimes SL_{\mathbb{Z}}(k)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Matrix of co-minors

$$(x_{ij}) \mapsto (f_{ij})$$

$$f_{ij} = (-1)^{i+j} \det(X \text{ leave out } i^{\text{th}} \text{ row} \times j^{\text{th}} \text{ col})$$

$$GL_{\mathbb{Z}}(1)(R) = \text{units in } R$$

$$A(x, y; xy-1) \text{ "hyperbola"}$$

$$\mu((x, y), (x', y')) = (xx', yy')$$

$$\iota((x, y)) = (y, x)$$

$$\frac{\mathbb{Z}[x, y]}{(xy-1)} \rightarrow R \quad \begin{array}{l} x \rightarrow \text{unit } u \\ y \rightarrow u^{-1} \end{array}$$

$$SL_{\mathbb{Z}}(2) = \mathbb{A}(a, b, c, d; ad - bc = 1)$$

$$O_{\mathbb{Z}}(2) = \mathbb{A}(a, b, c, d; a^2 + b^2 = 1, c^2 + d^2 = 1, ac + bd = 0)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B_{\mathbb{Z}}(2) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}; ad = 1 \right\}$$

$$\cap SL_{\mathbb{Z}}(2) \quad \boxed{\begin{array}{l} \text{Hypersurface } \times \mathbb{A}^1 \\ \text{in } \mathbb{A}^4 \\ G_m = GL_{\mathbb{Z}}(1) \end{array}}$$

but multiplication is extra  $\leftarrow$  in a space

$\mathbb{Z}$ -affine Algebraic groups

$$\left. \begin{array}{l} \mathbb{A}_{\mathbb{Z}}^1 \times \mathbb{A}_{\mathbb{Z}}^1 \xrightarrow{\alpha} \mathbb{A}_{\mathbb{Z}}^1 \\ (x, y) \xrightarrow{\alpha} (x+y) \\ (x, y) \xrightarrow{\mu} (xy) \\ \mathbb{A}_{\mathbb{Z}}^1 \xrightarrow{\nu} \mathbb{A}_{\mathbb{Z}}^1 \\ x \mapsto -x \end{array} \right\} \text{Morphisms.}$$

Functors (Ring  $\rightarrow$  Sets)

(Ring  $\rightarrow$  Group)

(Ring  $\rightarrow$  Ring)

$$G = O_{\mathbb{Z}}(2)$$

$G(R)$  is a group.  
for any commutative ring  $R$

$$O_{\mathbb{Z}}(2)(\mathbb{Z}/2\mathbb{Z})$$

finite group.

$$\begin{array}{ccc} 0 & \rightarrow & 0 \\ 1 & \rightarrow & 1 \\ \mathbb{Z} & \rightarrow & R \end{array}$$

$$SL_{\mathbb{Z}}(2)(\mathbb{Z}/n\mathbb{Z})$$

$$\mathbb{A}_{\mathbb{Z}}^1(R) = \text{Hom}(\mathbb{A}_{\mathbb{Z}}^1, R) = R$$

$$\mathbb{A}_{\mathbb{Z}}^1 = \mathbb{A}(x)$$

# Topology on $\mathbb{R}$ :

$$R = \underline{\mathbb{R}}, \underline{\mathbb{C}}, \mathbb{Q}_p,$$

$\mathbb{R}^n$  (induced) product topology

Polynomials are continuous functs.

"Topological Ring/Field"

$$\left( \begin{array}{c} \mathbb{R} \times \mathbb{R} \xrightarrow{d, \mu} \mathbb{R} \quad \mathbb{R} \xrightarrow{i} \mathbb{R} \\ \text{are continuous.} \end{array} \right)$$

Polynomials are continuous function.

$$\left( \begin{array}{c} \underline{X(\mathbb{R})}, \underline{X(\mathbb{C})}, \underline{X(\mathbb{Q}_p)} \\ X(\mathbb{R}_c) \end{array} \right)$$

Topological spaces!

"Strong" Topologies  $\rightarrow$  "more cont. functs."

CRing

$$\text{TopCRing} \xrightarrow{X} \text{Top}$$



Patching

$\mathbb{Z}$

# Prakam

Can we view Rings as functors.

$\mathbb{C}$  Ring  $\rightsquigarrow$  Set

$$R \rightsquigarrow X(R) = \text{Hom}(\underline{U(X)}, R)$$

$S$  is a ring

$$R \rightsquigarrow \text{Hom}(S, R) = S^*(R)$$

$$\begin{array}{ccc} R_1 \rightarrow R_2 & \rightsquigarrow & \text{Hom}(S, R_1) = S^*(R_1) \\ \downarrow \text{...} & & \downarrow \\ S \rightarrow R_1 & & \text{Hom}(S, R_2) = S^*(R_2) \end{array}$$

Any ring gives a functor  $\mathbb{C}\text{Ring} \rightsquigarrow \text{Set}$

Satisfies Patching.

$$\begin{array}{ccccc} S^*(R) & \xrightarrow{(f_i)} & S^*(R_{u_i}) & \xrightarrow{} & S^*(R_{u_i u_j}) \\ & & (k_i) & \xrightarrow{} & (g_{ij}, h_i \\ & & & & -g_{ji}, i(h_j)) \end{array}$$

$$\begin{array}{ccc} \text{hom: } S \xrightarrow{k_i} R_{u_i} & & \\ \downarrow & \supset & \downarrow \\ R_{u_j} & \xrightarrow{} & R_{u_i u_j} \end{array} \quad \langle u_1, \dots, u_n \rangle = \mathbb{Q} \\ \Rightarrow R \rightsquigarrow R \quad \underline{\sum u_i x_i = 1}$$

$S^*$  is the scheme "Spec(S)"  
is not  $\mathbb{Z}$ -affine

$$S = \mathbb{R}, \quad (\mathbb{R}[x, y] / \langle x^2 + y^2 - 1 \rangle)$$

$$S = \mathbb{C}, \quad \text{Spec} \\ \text{Spec}(\mathbb{R}), \text{Spec}(\mathbb{C})$$

$\mathbb{C}$  Ring  $\rightsquigarrow$  Set

which satisfy patching.

"local" condn:

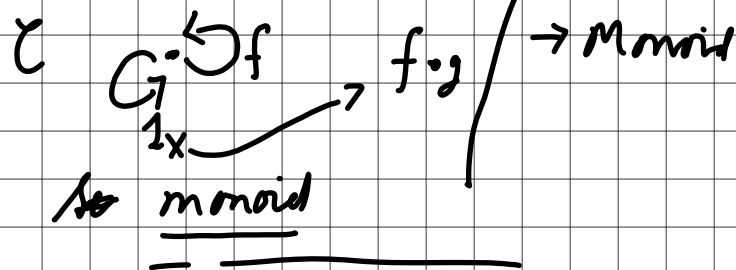
$G$  = groups and relations  
 = Multiplicative type  
 = axiomatic def.

Group as a category.

$\mathcal{C} \ni g$   
 Objects = Singleton  
 morphisms = group elements

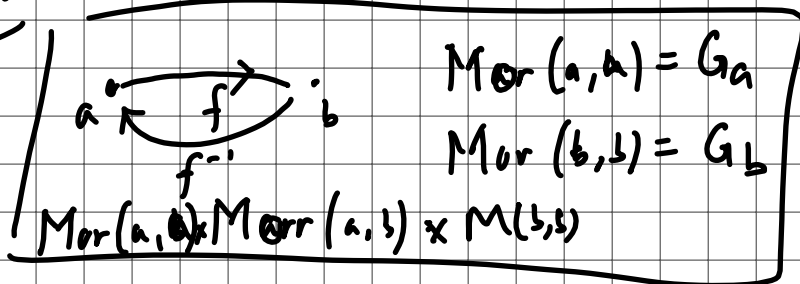
Group is a category  
 in which there is  
 one object &  
 every morphism is  
 an isomorphism.

Category with only one object.



Category in which every morphism is  
 an isomorphism.

Groupoid



Many definitions of the same  
 mathematical object

$\mathbb{R}$  - Dedekind cuts  
 - Cauchy sequences

$\mathbb{Q} \rightarrow \mathbb{Q}_p$   
 Schemes - "locally ring spaces"  
 Functors.

in Analysis most spaces are  
 closed subspaces of  $\mathbb{R}^n$   
 instead define in general  
 topological spaces

$\mathbb{R}^n$

Subset  $G$  of permutation group  $S_n$   
 Closed under multiplication  
 contains 1