Ayush Khare

When we have introduced projective space it is defined as a affine space quotienting by a equivalence relation, and then we have shown an affine space is sitting inside a projective space, how can we distinguished between the two notion, which one is more powerful, till now I am assuming projective space is something which has one less dimension corresponding to the affine plane.

First of all, it is good to avoid confusing affine space with a vector space even though they are *very* close.

In affine space there is no given origin. In a vector space there is a chosen origin. This is the only difference, but it is important. In particular, a vector space has a natural (additive) group structure and an action of scalar multiplication. An affine space has no uniqely defined additive group structure or scalar multiplication.

For the explanation of how affine space sits in projective space, see the answer to the question by Barnali Jana.

$\mathbb{A}^p$ = p-dim'l affine space (scheme)

$\underline{0} \in \underline{\mathbb{V}}^p$ = " vector space (scheme)

— pointed affine space

Ayush Khare

Can you elaborate the definition of Z- affine scheme why we need these extra polynomial f1, ...,fq are they the algebraic equations of transcendental numbers.

Z-affine schemes

1. $X = \mathbb{A}(\,;\,)$: $\mathcal{O}_X = \mathbb{Z}$ . . . o .

2. $X = \mathbb{A}(x_1, x_2, \dots, x_p;\,)$ Z-affine space of dim p

$\mathcal{O}_X = \mathbb{Z}[x_1, \dots, x_p]$

3. $X = \mathbb{A}(x_1, x_2;\ x_1^2 + x_2^2 - 1)$; $\mathcal{O}_X = \dfrac{\mathbb{Z}[x_1, x_2]}{\langle x_1^2 + x_2^2 - 1 \rangle}$

$\mathbb{R}[x,y]/\langle x^2+y^2-1\rangle \sim$ circle

$\mathbb{Z}$-circle!

4. $X = \mathbb{A}(x_1, x_2;\ x_1^2 + x_2^2 - 1, 5)$; $\mathcal{O}_X = \dfrac{\mathbb{Z}[x_1, x_2]}{(x_1^2 + x_2^2 - 1, 5)}$

$= \mathbb{F}_5[x_1, x_2]/(x_1^2 + x_2^2 - 1)$ — Circle/$\mathbb{F}_5$

Not "transcendental numbers". $\Leftrightarrow$ |variables|

Strictly speaking "transcendental numbers" (like π and e) do not play any special role in the basic theory of algebraic schemes. Thus, we can replace them with variables.

The equations $f_1,\dots,f_q$ are *part* of what defines an algebraic scheme. (The other part being the variables.)

We should think of $A(x_1,\dots,x_p;f_1,\dots,f_q)$ as the locus in p-dimensional affine space where the equations $f_i=0$ for $i=1,\dots,q$ are satisfied.

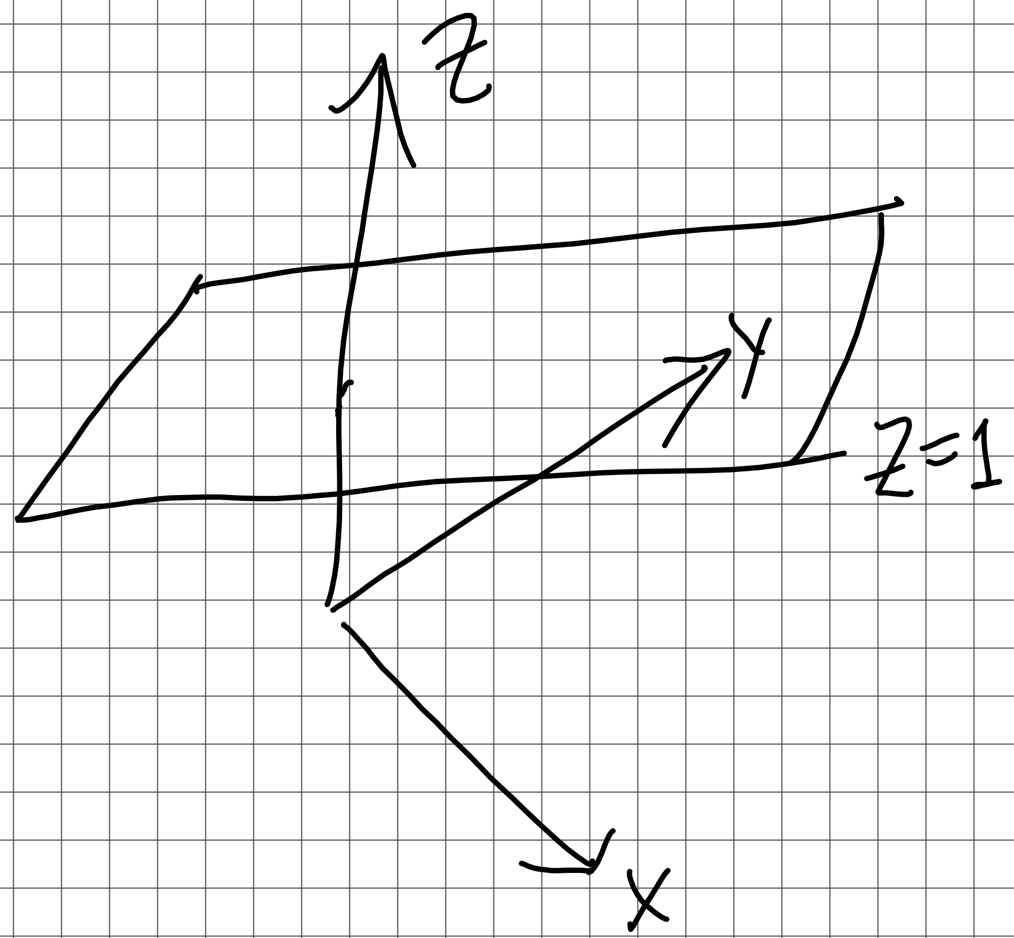
We will see some examples.

$\mathbb{A}(x_1, \dots, x_p; f_1, \dots, f_q)$

Barnali Jana

Anyone please explain the below line :

" If one enlarges $\C^{n}$ by adding infinity where parallel lines or asymptotic curves can be thought of as meeting ; the resulting space is called complex projective space ."



For a field F, we can think of affine n-space A as sitting inside the vector space F^{n+1} as the *affine* subspace given by $x_0=1$.

A line in A is given by its intersection with a 2-dimensional vector subspace of F^{n+1}

If P_1 and P_2 are two such 2-dimensional vector subspaces, then the corresponding lines are parallel if the intersection of P_1 and P_2 is a line that does not meet A.

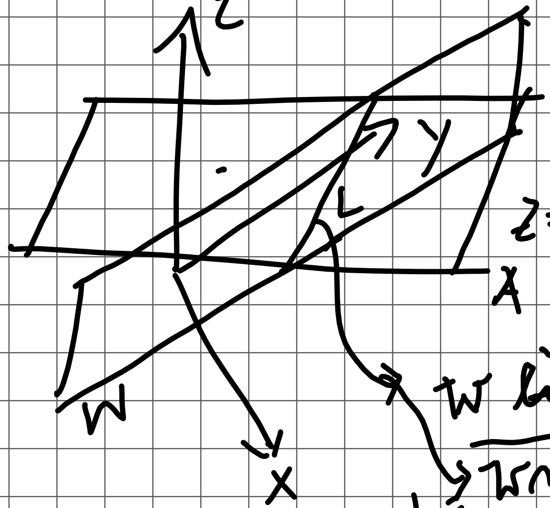
However, by our definition of projective space, each such line does give a point in projective space.

Thus we can see that lines that do not meet A have been *added* to the affine space in order to get projective space.

$0 \in V \supset A$ codim 1 - linear

does not contain

$0 \notin A$



$Z=2$

$Z=1$

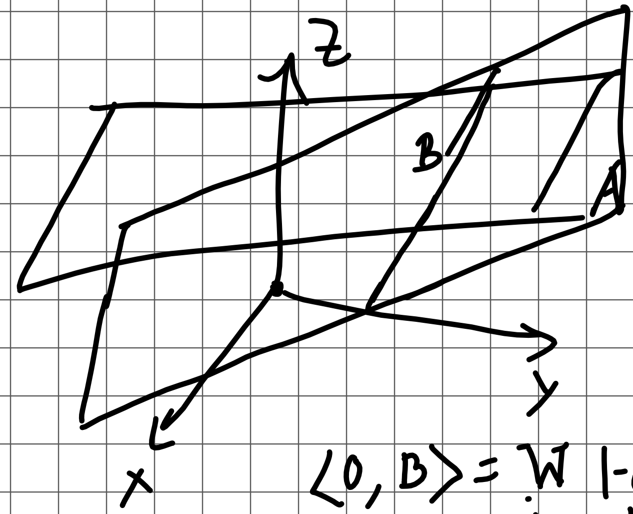
$0 \in V \supset A$

codim 1 space

$0 \notin A$

W ~~linear~~ vector sub space in V

$L = W \cap A$ is 1-dim line thru dim W



$\langle 0, B \rangle = W$ 1-dim line thru B.

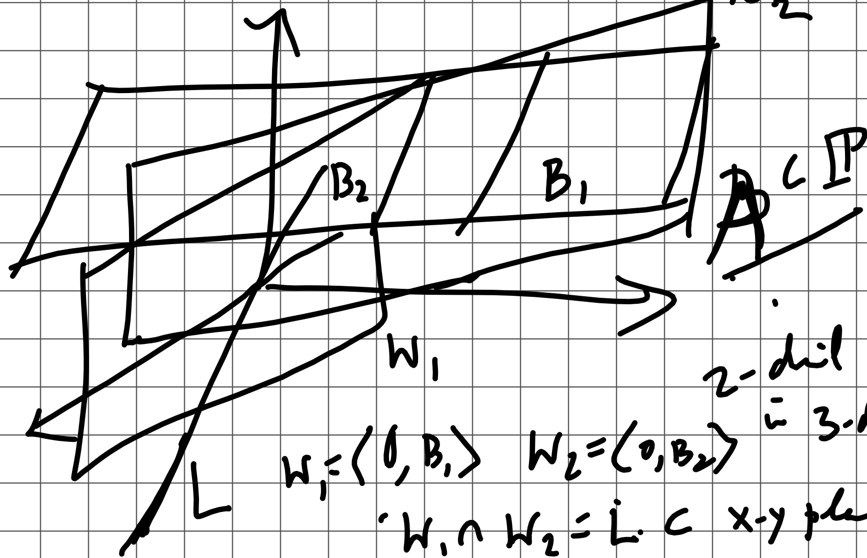
$W \cap A = B$

$(W \subset V$ vector subspace$) \longleftrightarrow ($linear subspace of $A)$

$L \subset W =$ xy-plane vector-space $\longleftrightarrow L \cap A = 0.$

$W \cap A = 0$

Hyperplane at ∞



$A \subset \mathbb{P}^2$

2-dim in 3-d space

$W_1 = \langle 0, B_1 \rangle \quad W_2 = \langle 0, B_2 \rangle$

$W_1 \cap W_2 = L \subset$ x-y plane

Vector subspaces of k^{p+1} $\longleftrightarrow$ linear projective subspaces of $\mathbb{P}^p$

W $\qquad\qquad$ $\mathbb{P}(W)$

$k\cdot(X_0, \dots, X_p)$ $\longleftrightarrow$ $[X_0 : \dots : X_p]$

x

$(X_0 = 1)$ $\qquad$ A^p $\qquad\qquad$ $X_0 = 0$

$k^{p+1} \setminus \{0\}$

Safarevic

Hodge - Pedoe. $k \setminus \{0\}$

$\mathbb{P}^r(\mathbb{F}_q)$ $\quad q = p^r$ = prime power.

$$\left(\mathbb{F}_q^{(r+1)} \setminus \{0\}\right) \Big/ \left(\mathbb{F}_q \setminus \{0\}\right) \qquad \frac{(q)^{r+1}-1}{(q-1)} = \boxed{1+q+\dots+q^r}$$

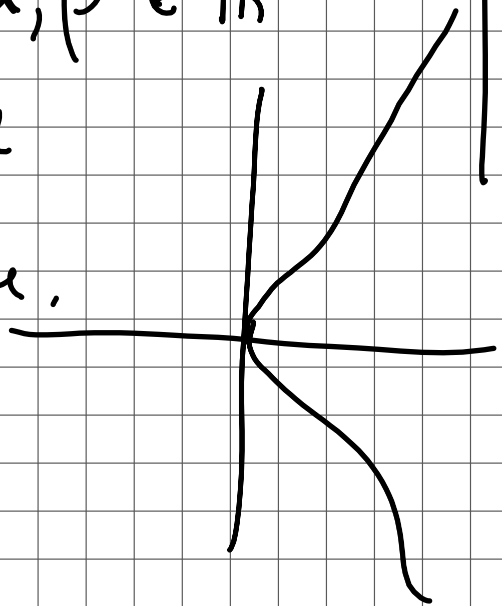
$$\mathbb{P}^{r-1}(\mathbb{F}_q) \subset \mathbb{P}^r(\mathbb{F}_q) \supset \underline{\mathbb{A}^r(\mathbb{F}_q)} \quad q^r$$

$$A(x,y;\, y^2 - x^3 - x) = E$$

$E(\mathbb{R})$ $\qquad \alpha, \beta \in \mathbb{R}$

$$\beta^2 = \alpha^3 + \alpha$$

plane curve.



$a = 2,\ b = 3$

$$\mathcal{O}_X = \mathbb{Z}[x,y] \big/ \langle 2x^2 + 3y^2 - 1 \rangle$$

$$X = A(x,y;\, 2x^2 + 3y^2 - 1)$$

$X(\mathbb{R})$ $\qquad \mathbb{Z}[x,y]/\langle\ \rangle \longrightarrow \mathbb{R}$, $\quad \langle\ \rangle \longrightarrow 0$

$x \longrightarrow \alpha$

$y \longrightarrow \beta$

$$\underline{2\alpha^2 + 3\beta^2 - 1 = 0}$$

$X(\mathbb{R})$ is an ellipse.

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{Q} \right\} = a \cdot 1 + b n$$

$$= \mathbb{Q} \oplus \mathbb{Q} n \qquad n^2 = 0 \qquad n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\frac{\mathbb{Z}_1 [x, y]}{\langle 2x^2 + 3y^2 - 1 \rangle} \longrightarrow S$$

$$x \longmapsto \alpha_1 + \alpha_2 n$$

$$y \longmapsto \beta_1 + \beta_2 n$$

$$2(\alpha_1 + \alpha_2 n)^2 + 3(\beta_1 + \beta_2 n)^2 - (1 + 0 \cdot n) = 0$$

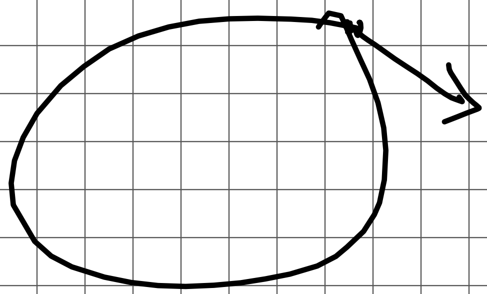
$$2\alpha_1^2 + 4\alpha_1\alpha_2 n + 3\beta_1^2 + 6\beta_1\beta_2 n - 1 + 0 \cdot n = 0$$

$$2\alpha_1^2 + 3\beta_1^2 - 1 = 0$$

(α_1, β_1) is a point of the ellipse.

$$4\alpha_1\alpha_2 + 6\beta_1\beta_2 = 0$$

(α_2, β_2) is a tangent vector to ellipse at (α_1, β_1)



Grothendieck's. $\longleftrightarrow$ look at R-valued solutions.

$$\boxed{X = \mathbb{A}(x_1, \dots, x_p ; f_1, \dots, f_q)} \leftarrow \text{DEFN}$$

$$\mathcal{O}_X = \mathbb{Z}[x_1, \dots, x_p] / \langle f_1, \dots, f_q \rangle$$

$$\Updownarrow$$

Any finitely generated commutative ring with identity