Linear varieties

In high-school we learn to solve systems of linear equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,p}x_p &= c_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,p}x_p &= c_2 \\ &\vdots \\ a_{q,1}x_1 + a_{q,2}x_2 + \cdots + a_{q,p}x_p &= c_q \end{aligned}$$

where $a_{i,j}$ lie in a field k.

The solutions are found by converting this to the matrix form:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,p} & c_1 \\ a_{2,1} & a_{2,2} & \ldots & a_{2,p} & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{q,1} & a_{q,2} & \ldots & a_{q,p} & c_q \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Note the -1 which brings the column of c_i's into the matrix!

One then performs row reduction on the matrix to reduce it into