Steiner's "Algebra of Throws"

MTH437 — Introduction to Schemes

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Recall

The projective space $\mathbb{P}^p(k)$ was introduced as the collection of equivalence classes of points $[x_0 : x_1 : \cdots : x_p]$ where:

- The vector $(x_0, x_1, \ldots, x_p)$ is a non-zero vector in k^{p+1}.
- Two vectors which are multiples of each other give the same point.

Projective linear subspaces of $\mathbb{P}^p(k)$ are of the form $\mathbb{P}(W)$ which consists of those points which are associated with vectors in a vector subspace W of k^{p+1}.

When W is a $d+1$-dimensional space, $\mathbb{P}(W)$ is d-dimensional and vice versa.

There is a natural “dictionary” between linear algebra and the study of projective linear subspaces. The important properties are as follows.

- Given a point $\mathbb{P}(L)$ of $\mathbb{P}^p(k)$ and a linear subspace $\mathbb{P}(W)$ of $\mathbb{P}^p(k)$ that does not contain $\mathbb{P}(L)$, the join (or span) $\mathbb{P}(L+W)$ is a linear subspace of $\mathbb{P}^p(k)$ that has dimension 1 more than that of $\mathbb{P}(W)$ and it contains $\mathbb{P}(L)$ and $\mathbb{P}(W)$.
- Given two subspaces $\mathbb{P}(V)$ and $\mathbb{P}(W)$ of $\mathbb{P}^p(k)$ of dimensions a and b respectively; note that this means that dimensions of V and W are $a+1$ and $b+1$ respectively. If $a+b \geq p$, then since the dimension $V \cap W$ is at least $(a+1)+(b+1)-(p+1) \geq 1$. Hence, $\mathbb{P}(V) \cap \mathbb{P}(W) = \mathbb{P}(V \cap W)$ is a linear space of dimension at least $a+b-p$; in particular, it is non-empty.

Such observations allow us to answer a number of simple questions about linear spaces in projective space.

In particular, we were able to solve the following:

Question: Given three lines $\mathbb{P}(A)$, $\mathbb{P}(B)$ and $\mathbb{P}(C)$ in projective space $\mathbb{P}^3(k)$, is there a line that meets all three lines?

A similar looking problem turns out to be rather more complicated.

Question: Given four lines A, B, C and D in projective space $\mathbb{P}^3(k)$, is there a line that meets all four lines?

This apparently linear question leads to non-linear (quadratic) equations in terms of coordinates.

Algebra is emergent!

Solving questions involving configurations of linear spaces *forces* us to consider equations of higher degree.

This can be seen as a consequence of the fact that addition and multiplication of coordinates *arises* out of configurations of lines as explained below.

In other words, to do geometry, we must do algebra!

Addition

Given the points $p = (1 : 0 : a)$ and $q = (1 : 0 : b)$ on the line A given by $x_1 = 0$. We will show that $(1 : 0 : a + b)$ arises when we examine the point of intersection of this line with another naturally arising line. To do this we also need the "origin" $o = (1 : 0 : 0)$ which represents the identity element for addition. Note that $r = (0 : 0 : 1)$ is the "point at infinity" on the line A since the line B given by $x_0 = 0$ is the line at infinity on the plane.

- Consider the line C given by $x_1 = x_0$. The lines A, B and C meet in the point $r = (0 : 0 : 1)$. Since B is the line at infinity, we may think of A and C as parallel lines.
- Consider the point $s = (0 : 1 : 0)$ which is on B but not on A or C. Since B is the line at infinity we can think of s as a "direction" *different* from that of A and C (which are parallel).
- We have the line D which joins p and s. This is a line through p "in the direction" given by s. This meets C in some point t.
- We have the line E which joins o and t. This meets B in some point u. Then u represents the direction of the line E.
- We have the line F which joins q and u. This is the line through q which is in the direction given by u. In other words, it is through q and parallel to E. This meets C in some point v.
- We have the line G which joins v and s. This is the line through v which is parallel to D. This meets A in some point w.

The claim is that the coordinate of w is $(1:0:a+b)$. Let us work this out!

1. The line D is given by $x_2 = ax_0$ since p and s satisfy this equation.
2. This means that the point t is given by $(1:1:a)$.
3. Thus the line E is given by $x_2 = ax_1$ since o and t satisfy this equation.
4. This means that the point u is given by $(0:1:a)$.
5. Thus the line F is given by $x_2 = ax_1 + bx_0$ since u and q satisfy this equation.
6. This means that the point v is given by $(1:1:a+b)$.
7. Thus the line G is given by $x_2 = (a+b)x_0$ since s and v satisfy this equation.
8. This means that the point w is given by $(1:0:a+b)$ as required.

- Note that C could have been *any* line passing through the point r other than A. In other words, it could be any line parallel to A.
- Note that s could have been *any* point on B other than r. In other words, it could be any direction other than that of A.

Multiplication

Given the points $p = (1 : 0 : a)$ and $q = (1 : 0 : b)$ on the line A given by $x_1 = 0$. We will show that $(1 : 0 : a \cdot b)$ arises when we examine the point of intersection of this line with another naturally arising line. To do this we also need the "origin" $o = (1 : 0 : 0)$ and the point $i = (1 : 0 : 1$ which represents the identity element for multiplication. Note that $r = (0 : 0 : 1)$ is the "point at infinity" on the line A since the line B given by $x_0 = 0$ is the line at infinity on the plane.

As before, we need to choose some additional lines and points.

- Consider the line C given by $x_2 = 0$. The lines B and C meet in the point $s = (0 : 1 : 0)$.
- Consider the point $t = (1 : 1 : 0)$ which is on C but not on A or B.
- We have the line D that joins i and t. This meets the line B at a point u.
- We have the line E that joins p and u. This meets the line C at a point v.
- We have the line F that joins q and t. This meets the line B at the point w.
- We have the line G that joins v and w. This meets the line A at the point x.

The claim is that this point x is $(1 : 0 : a \cdot b)$. Let us verify this.

1. Note that the line D is given by $x_1 + x_2 = x_0$ since both i and t lie on this line. This means that $u = (0 : 1 : -1)$.
2. Note that the line E is given by $x_1 + x_2 = ax_0$ since both u and p lie on this line. This means hat $v = (1 : a : 0)$.
3. Note that the line F is given by $bx_1 + x_2 = bx_0$ since both q and t lie on this line. This means hat $w = (0 : 1 : -b)$.
4. Note that the line G is given by $bx_1 + x_2 = abx_0$ since both v and w lie on this line. This means that $x = (1 : 0 : ab)$ as required.

Conclusion

- In order to study linear geometry, it is convenient to work with projective spaces so that rank is the only thing that determines consistence.
- The geometry of linear projective varieties is closely related to the corresponding vector subspaces.
- The intersection and join of linear projective varieties can be easily understood in terms of geometric ideas based on two notions:
  - Every pair of distinct points determines a line.
  - Every pair of linear projective subspaces of dimensions a and b in $\mathbb{P}^p(k)$ intersects in a linear projective space of dimension $a + b - p$ *provided* $a + b \geq p$.
- One can pose problems of finding certain configurations of linear projective varieties. Such problems *naturally* lead to problems in algebra that require the solutions of polynomial equations of all degrees.