Linear Subspaces

MTH437 — Introduction to Schemes

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Linear varieties

In high-school we learn to solve systems of linear equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,p}x_p &= c_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,p}x_p &= c_2 \\ &\vdots \\ a_{q,1}x_1 + a_{q,2}x_2 + \cdots + a_{q,p}x_p &= c_q \end{aligned}$$

where $a_{i,j}$ lie in a field k.

The solutions are found by converting this to the matrix form:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,p} & c_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,p} & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{q,1} & a_{q,2} & \cdots & a_{q,p} & c_q \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Note the -1 which brings the column of c_i's into the matrix!

One then performs row reduction on the matrix to reduce it into row-echelon form.

One important aspect of this procedure is that it *does* not change the solutions.

We observe the following with the row-echelon form:

1. We can eliminate all rows that are identically 0.
2. If there is any row that looks like $(0\ 0\ \cdots\ 0\ c)$ where c is non-zero then the system of equations is *inconsistent*. There is *no* solution in this case.

When (2) does not happen, we say that our system of linear equations is *consistent*. In this case, there *are* solutions. What does the locus of solutions look like?

Affine linear subspaces

Assuming that the system of equations is *consistent*, we are left with r *independent* linear equations in p unknowns. The locus L of solutions in k^p is an *affine linear subspace* of dimension $p - r$.

Affine linear subspace of k^p of dimension d: A subset L of k^p is an *affine linear subspace* of dimension d if there is a d-dimensional vector subspace V of k^p such that for any point p in L, we have $L = p + V$.

In particular, a single non-trivial (not all coefficients of x_i's are 0) linear equation defines an affine linear subspace of dimension $p - 1$. This is also called an *affine hyperplane*.

We can this think of L as an *intersection* of q affine hyperplanes (only r of which give independent conditions).

When $c \neq d$, the hyperplanes H_c and H_d defined by their respective equations

$$a_1 x_1 + a_2 x_2 + \cdots + a_p x_p = c$$
$$a_1 x_1 + a_2 x_2 + \cdots + a_p x_p = d$$

do *not* intersect. (We sometimes say that H_c and H_d are *parallel*.) This leads to inconsistent systems as can be seen easily.

Exercise: Find a system of linear equations that are *pairwise* consistent, but the totality of the system is inconsistent. (Hint: We need $p \geq 3$.)

Projective space

The occurrence of inconsistent equations is annoying as we would like to treat all matrices of rank r on equal footing.

One solution is to add a new variable x_0 and write the equations in *homogenised* form as:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,p}x_p &= c_1 x_0 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,p}x_p &= c_2 x_0 \\ &\vdots \\ a_{q,1}x_1 + a_{q,2}x_2 + \cdots + a_{q,p}x_p &= c_q x_0 \end{aligned}$$

Now, an "inconsistent" equation becomes simply $0 = cx_0$ (with $c \neq 0$).

The solutions of this system of equations forms a *vector subspace* W of k^{p+1}. In fact, if the matrix above has rank r, then this subspace has dimension $p + 1 - r$.

One issue to note is that if $(x_0, x_1, \ldots, x_p)$ is a solution, then so is $(dx_0, dx_1, \ldots, dx_p)$, for any element d in k. This should not be seen as a "new" solution.

We thus introduce the *projective space* $\mathbb{P}^p(k)$ as the collection of all non-zero tuples $(x_0, x_1, \ldots, x_p)$ where two tuples are considered equivalent if they are multiples of each other by an element of the field. We denote the equivalence class by $(x_0 : x_1 : \cdots : x_p)$.

Projective linear subspace of $\mathbb{P}^p(k)$: A *projective linear subspace* of $\mathbb{P}^p(k)$ of dimension d is precisely the locus of equivalence classes of points associated with a vector subspace W of k^{p+1} of dimension $d+1$; it is usually denoted by $\mathbb{P}(W)$ or $\mathbb{P}(W)(k)$.

Giving a basis for this (finite dimensional) vector space W gives a *bijection* between $\mathbb{P}(W)$ and $\mathbb{P}^d(k)$.

If W has dimension p, then $\mathbb{P}(W)$ is a $p-1$ dimensional linear subspace called a *projective hyperplane* in $\mathbb{P}^p(k)$.

Hyperplane at "infinity"

If $(x_0 : x_1 : \cdots : x_p)$ is a point of $\mathbb{P}^p(k)$ with $x_0 \neq 0$, then this is equal to the point $(1 : y_1 : \cdots : y_p)$ where $y_i = x_i/x_0$.

Thus, if H_0 is the subspace of $\mathbb{P}^p(k)$ corresponding to the subspace of k^{p+1} defined by $x_0 = 0$, then its *complement* $U_0 = \mathbb{P}^p(k) \setminus H_0$ can be identified with k^p in a natural way.

This allows us to *identify* the solution locus L for a *consistent* system of linear equations as studied above with $U_0 \cap \mathbb{P}(W)$ where W is the solution vector space for the homogenised system of equations. Note that consistence ensures that W is not entirely contained in the subspace defined by $x_0 = 0$; in particular, this means W is not the zero subspace!

Under a change of basis of k^{p+1}, the equation $x_0 = 0$ looses its "special" significance. In fact, the general linear group operates transitively on $k^{p+1} \setminus \{0\}$. As a consequence, we see that $\mathbb{P}^p(k)$ carries a transitive action of this group as well.

Linear Algebra

The study of (projective) linear subspaces of projective space $\mathbb{P}^p(k)$ is entirely captured by the study of vector subspaces of k^{p+1}.

- A 1-dimensional vector subspace L of k^{p+1} gives a *point* $\mathbb{P}(L)$ in $\mathbb{P}^p(k)$; it corresponds to a non-zero vector *up to* scalar multiple.
- Distinct points of $\mathbb{P}^p(k)$ correspond to linearly independent vectors.
- Given two vector subspaces V and W of k^{p+1}, their intersection $V \cap W$ is a vector subspace. Thus $\mathbb{P}(V)$ and $\mathbb{P}(W)$ intersect in $\mathbb{P}(V \cap W)$ *provided* this intersection is non-zero.
- The span $V + W$ of vector subspaces gives $\mathbb{P}(V + W)$ which is the *join* of $\mathbb{P}(V)$ and $\mathbb{P}(W)$. It is is the smallest linear subspace of $\mathbb{P}^p(k)$ that contains both of them.

- The join of two distinct points $\mathbb{P}(L)$ and $\mathbb{P}(M)$ (where L and M are 1-dimensional vector subspaces) is $\mathbb{P}(L+M)$ which is a 1-dimensional linear subspace of $\mathbb{P}^p(k)$ (since $L+M$ is a 2-dimensional vector subspace). It is called the *projective line* joining the points. *Two points determine a line.*
- A non-zero linear functional on k^{p+1} determines, via its kernel, a projective hyperplane in $\mathbb{P}^p(k)$. Conversely, such a hyperplane determines a non-zero linear functional up to scalar multiple.

Such observations allow us to answer a number of simple questions about linear spaces in projective space.

Lines meeting three lines in space

Question: Given three lines $\mathbb{P}(A)$, $\mathbb{P}(B)$ and $\mathbb{P}(C)$ in projective space $\mathbb{P}^3(k)$, is there a line that meets all three lines?

Note that A, B and C are *two* dimensional vector subspaces of k^4. We will assume that $A \cap B = B \cap C = C \cap A = \{0\}$ as the "degenerate" cases where they meet in non-zero subspaces are easier.

- Pick a non-zero vector v in C.
- Since $A \cap C = \{0\}$, we see that the vector space $D = A + k \cdot v$ is 3-dimensional. Similarly, $E = B + k \cdot v$ is also 3 dimensional.
- Now D and E are vector subspaces of k^4, so it follows that $D \cap E$ has dimension at least $3 + 3 - 4 = 2$. Note that $D \cap E$ contains $k \cdot v$ as well.

- Suppose F is a 2-dimensional vector subspace of $D \cap E$ which contains v.
- We note that $\mathbb{P}(F)$ is a line in $\mathbb{P}^3(k)$ which contains $\mathbb{P}(k \cdot v)$ which is a point of $\mathbb{P}(C)$.
- Moreover, A and F are a 2-dimensional subspaces of D which is a 3-dimensional space. It follows that $A \cap F$ has dimension at least $2 + 2 - 3 = 1$.
- In other words, $\mathbb{P}(A) \cap \mathbb{P}(F) = \mathbb{P}(A \cap F)$ is non-empty. Similarly, $\mathbb{P}(B) \cap \mathbb{P}(F)$ is non-empty.

Thus, we see that $\mathbb{P}(F)$ is a line in $\mathbb{P}^3$ that meets the lines $\mathbb{P}(A)$, $\mathbb{P}(B)$ and $\mathbb{P}(C)$.

One can give the same argument a bit more "projectively". First of all we note two things about linear subspaces of $\mathbb{P}^p(k)$.

- Given a point $\mathbb{P}(L)$ of $\mathbb{P}^p(k)$ and a linear subspace $\mathbb{P}(W)$ of $\mathbb{P}^p(k)$ that does not contain $\mathbb{P}(L)$, the join (or span) $\mathbb{P}(L+W)$ is a linear subspace of $\mathbb{P}^p(k)$ that has dimension 1 more than that of $\mathbb{P}(W)$ and it contains $\mathbb{P}(L)$ and $\mathbb{P}(W)$.
- Given two subspaces $\mathbb{P}(V)$ and $\mathbb{P}(W)$ of $\mathbb{P}^p(k)$ of dimensions a and b respectively; note that this means that dimensions of V and W are $a+1$ and $b+1$ respectively. If $a+b \geq p$, then since the dimension $V \cap W$ is at least $(a+1)+(b+1)-(p+1) \geq 1$. Hence, $\mathbb{P}(V) \cap \mathbb{P}(W) = \mathbb{P}(V \cap W)$ is a linear space of dimension at least $a+b-p$; in particular, it is non-empty.

The above argument can be now be stated in terms of points, lines and planes in $\mathbb{P}^3$.

- Given three distinct projective lines A, B and C in $\mathbb{P}^3$ we want to find a projective line that meets all of them.
- Choose a point v of C that is not on A or B. (Since the lines are distinct, this is possible!)
- Let D be the projective plane joining v and A. Similarly, let E be the projective plane joining v and B.
- We note that $F = D \cap E$ in $\mathbb{P}^3$ has dimension at least $2 + 2 - 3 = 1$. On the other hand, since A and B are distinct its dimension cannot be more than 1!
- It follows that F is a line in $\mathbb{P}^3$ that contains v. Hence $F \cap C$ is non-empty.
- Since F and A are lines in the plane D, we see that $F \cap A$ must be non-empty as above. Similarly, $F \cap B$ is non-empty.

Lines meeting four lines in space

We see that there was a *choice* made in order to find the line F. This was the choice of v in C. Thus, we may expect that there is a "one parameter locus" of lines in space that meets a given collection of 3 distinct lines. As a consequence we *could* expect to solve the following:

Question: Given four lines A, B, C and D in projective space $\mathbb{P}^3(k)$, is there a line that meets all four lines?

The answer to this question turns out to *depend* on the field that we are looking at! In particular, we map need to solve a *quadratic* equation over the field k in order to find such a line.