Revision of Sequences

(3 marks) 1. Consider the following sequences

(a) The sequence $(a_n)_{n\geq 1}$ with

$$a_n = n^3 \cdot (3/4) - 25n^2$$

(b) The sequence $(b_n)_{n\geq 1}$ with

$$a_n = n^3 \cdot (3/5) + 50 \cdot n^2 + 10n$$

(c) The sequence $(c_n)_{n\geq 1}$ with

$$c_n = \frac{3^n - 100 \cdot n^4}{2^n \cdot 500}$$

Which sequence dominates which sequence for large values of n?

2. Which of the following series of positive terms converges. Justify your answer by giving an upper bound for the sum.

(1 mark) (a) The series $\sum_{k=1}^{\infty}(6/7)^k$

(1 mark) (b) The series $\sum_{k=0}^{\infty} 1/(2k+1)$

(1 mark) (c) The series $\sum_{k=2}^{\infty} 1/(k^2-1)$

(3 marks) 3. Show that the following sequence is *increasing* and bounded above and below: $x_1 = 1$ and

$$x_{n+1} = \frac{13x_n + 13}{x_n + 13}$$

(Hint: Compare x_n^2 with 13.)

4. Define the sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$, $(z_n)_{n\geq 1}$ as follows. First of all we have the identity

$$z_n = \frac{2x_n + 3y_n}{5}$$

We define $x_1 = 2$ and $y_1 = 0$. Finally, x_n and y_n are defined defined as:

1. If $z_n^3 \leq 5$, then $y_{n+1} = z_n$ and $x_{n+1} = x_n$.
2. If $z_n^3 > 5$, then $y_{n+1} = y_n$ and $x_{n+1} = z_n$.

(1 mark) (a) Show that $(a+b)/2$ lies between a and b.

(1 mark) (b) Show by induction that $x_n > z_n > y_n$ for all n.

(1 mark) (c) Show that $(x_n)_{n\geq 1}$ is an increasing sequence and $(y_n)_{n\geq 1}$ is a decreasing sequence.

(1 mark) (d) Show by induction that $(x_n - y_n) = 1/2^{n-2}$.

(1 mark) (e) Show the following inequalities.

$$\limsup(z_n)_{n\geq 1} \geq \sup(y_n)_{n\geq 1}$$
$$\liminf(z_n)_{n\geq 1} \geq \inf(x_n)_{n\geq 1}$$

(1 mark) (f) Show that all three sequences converge and have the same limit.