Analysis in One Variable
MTH102 First Mid-Sem Exam 5 February 2019

Solutions

(3 marks) 1. Order the following sequences according *eventual* size (for large values of n).

(a) The sequence with general term $n^2 \cdot (2/3)$

$$2/3, 8/3, 6, 32/3, \ldots$$

(b) The sequence with general term $n^2 \cdot (3/5) + 100 \cdot n$

$$503/5, 1012/5, 1527/5, 2048/5, \ldots$$

(c) The sequence with general term $2^n \cdot (1/500) - 100 \cdot n$

$$-24999/250, -24999/125, -37498/125, -49996/125, \ldots$$

Solution: As seen in the notes, $(1+x)^n$ dominates any polynomial function of n for any $x > 0$. Secondly,

$$n^2(2/3) - n^2(3/5) - 100 \cdot n = n^2(1/10) - 100 \cdot n$$

Since n^2 dominates any an for any fixed n, we see that the order (for n sufficiently large is):

$$(2^n \cdot (1/500) - 100 \cdot n) > \left(n^2\,(2/3)\right) > \left(n^2(3/5) + 100 \cdot n\right)$$

1 Mark for each correct inequality and 1 Mark for justification.

2. Which of the following sequences is bounded above. If it is bounded above, give *an* upper bound.

(2 marks) (a) The sequence with general term $\sum_{k=1}^{n}(4/5)^k$

$$4/5, 36/25, 244/125, 1476/625, 8404/3125, \ldots$$

Solution: By the sum of the geometric series, we have

$$\sum_{k=1}^{n}(4/5)^k = \frac{4}{5} \cdot \frac{1-(4/5)^{k+1}}{1-(4/5)} = 4\left(1-(4/5)^{k+1}\right) < 4$$

1 Mark for correct bound. 1 Mark for justification.

(2 marks) (b) The sequence with general term $\sum_{k=1}^{n} 1/(5k)$

$$1/5, 3/10, 11/30, 5/12, \ldots$$

Solution: By the examination of the harmonic series in the notes, we have seen that

$$\sum_{k=1}^{2^n}(1/5k) = (1/5)\sum_{k=1}^{2^n}(1/k) \geq (1/5)\,(1 + (1/2)\,n)$$

Thus, this sequence is unbounded.

1 Mark for unboundedness. 1 Mark for justification.

(2 marks) (c) The sequence with general term $\sum_{k=1}^{n} 1/(2k^2)$

$$1/2, 5/8, 49/72, 205/288, \ldots$$

Solution: By the examination of the square harmonic series in the notes, we have seen that

$$\sum_{k=1}^{2^n-1}(1/2k^2) = (1/2)\sum_{k=1}^{2^n-1}(1/k^2) < (1/2)\sum_{k=0}^{n}(1/2^k) < 1$$

1 Mark for correct bound. 1 Mark for justification.

(3 marks) 3. Show that the following sequence is *decreasing* and bounded below: $x_1 = 4$ and

$$x_{n+1} = \frac{4x_n + 11}{x_n + 4}$$

(Hint: Compare x_n^2 with 11.)

Solution: We note that $x_1^2 > 11$. We wish to show by induction that $x_n^2 > 11$. Assuming this for a given n, we calculate

$$x_{n+1}^2 - 11 = \frac{(4x_n + 11)^2 - 11(x_n + 4)^2}{(x_n + 4)^2} = \frac{5x_n^2 - 55}{(x_n + 4)^2} > 0$$

This shows, by the principle of induction, that $x_n^2 > 11$ for all n. 1 Mark for this proof.

Since $x_n > 0$ for all n (also by induction!), it follows that $x_n > 3$ for all n. 1 Mark for a correct bound.

We calculate

$$x_n - \frac{4x_n + 11}{x_n + 4} = \frac{x_n^2 - 11}{x_n + 4} > 0$$

It follows that $x_n > x_{n+1}$ for all n. 1 Mark for this proof.

4. We define $x_1 = 1$ and $y_1 = 2$. We then iteratively define $z_n = (x_n + y_n)/2$ for $n \geq 1$, where x_n and y_n are defined as:

   1. If $z_n^3 \leq 7$, then $x_{n+1} = z_n$ and $y_{n+1} = y_n$.
   2. If $z_n^3 > 7$, then $x_{n+1} = x_n$ and $y_{n+1} = z_n$.

(1 mark) (a) Show that $(y_n - x_n)_{n\geq 1}$ is a decreasing sequence with greatest lower bound 0.

Solution: We note that $y_1 - x_1 = 1$. We also note that $z_1 - x_1 = 1/2$ and $y_1 - z_1 = 1/2$. By induction, we claim that

$$y_n - x_n = 1/2^n \ ; \ z_n - x_n = 1/2^{n+1} \ ; \ y_n - z_n = 1/2^{n+1}$$

Let us assume this for a given n. We note that in the case(1) above, we have

$$(x_{n+1}, y_{n+1}) = (z_n, y_n)$$

In the case(2) above, we have

$$(x_{n+1}, y_{n+1}) = (x_n, z_n)$$

So the first equality is obtained. Now, z_{n+1} is the mid-point of x_{n+1} and y_{n+1} so the next two inequalities follow as well. It follows that $y_n - x_n$ is decreasing to 0.

(1 mark) (b) Show *one* of the following inequalities. (The other one is similar!)

$$\limsup(z_n)_{n\geq 1} \geq \limsup(x_n)_{n\geq 1}$$
$$\liminf(z_n)_{n\geq 1} \leq \liminf(y_n)_{n\geq 1}$$

(The question had $\geq$ in the second line. Bonus for the students who pointed it out!)

Solution: We have seen above that $z_n - x_n > 0$. This shows that $z_n \geq x_n$. It follows that $\sup(z_n)_{n\geq k} \geq \sup(x_n)_{n\geq k}$. Hence,

$$\limsup(z_n)_{n\geq 1} = \inf(\sup(z_n)_{n\geq k})_{k\geq 1} \geq \inf(\sup(x_n)_{n\geq k})_{k\geq 1} = \limsup(x_n)_{n\geq 1}$$

The argument for z_n and y_n is similar.

(1 mark) (c) Show that $(z_n)_{n\geq 1}$ converges.

Solution: We have $y_n - x_n = (1/2^n)$. Hence,

$$\begin{aligned} \liminf(y_n)_{n\geq 1} &\leq \limsup(y_n)_{n\geq 1} \\ &= \limsup(x_n + (1/2^n))_{n\geq 1} = \limsup(x_n)_{n\geq 1} + 0 \\ &= \limsup(x_n)_{n\geq 1} \end{aligned}$$

Thus,

$$\begin{aligned} \liminf(z_n)_{n\geq 1} &\leq \liminf(y_n)_{n\geq 1} \leq \limsup(x_n)_{n\geq 1} \\ &\leq \limsup(z_n)_{n\geq 1} \leq \liminf(z_n)_{n\geq 1} \end{aligned}$$

Hence, we have equality as required.