## Solutions to Assignment 3

1. Which of the following series is convergent and which diverges to infinity?

(1 mark) (a) $\sum_{n=1}^{\infty} \frac{1}{n+20}$

**Solution:** We note that this series is

$$\frac{1}{21} + \frac{1}{22} + \cdots$$

So it is the same as

$$-\sum_{n=1}^{20} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n}$$

The first term above is a fixed constant since it has finitely many terms. The second is a series that diverges to infinity. So the sum of the two terms also diverges to infinity.

(1 mark) (b) $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$

**Solution:** We note that

$$\frac{n+1}{n^2} > \frac{1}{n}$$

It follows that

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2} > \sum_{n=1} \frac{1}{n}$$

Since the right-hand side diverges to infinity, so does the left-hand side.

(1 mark) (c) $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ for $0 < x < 1$.

**Solution:** We note that

$$\frac{x^n}{n+1} \leq x^n$$

It follows that

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} \leq \sum_{n=0}^{\infty} x^n$$

Since the right-hand side converges, so does the left-hand side.

(1 mark) (d) $\sum_{n=1}^{\infty} x^{n-n^2}$ for $0 < x < 1$

**Solution:** We note that, for $0 < x < 1$, and $m \geq 0$, we have $x^m \leq 1$. Now, $n^2 \geq n$ for $n \geq 1$, so $x^{n^2-n} \leq 1$. It follows that $x^{n^2} \leq x^n$, and equivalently

$x^{n-n^2} \geq 1$. Hence,

$$\sum_{n=0}^{\infty} x^{n-n^2} \geq \sum_{n=0}^{\infty} 1$$

Since the right-hand side diverges to infinity, so does the left-hand side.

(1 mark) (e) $\sum_{n=1}^{\infty} x^{n^2-n}$ for $0 < x < 1$

**Solution:** We have, for $n \geq 2$, $n - 1 \geq 1$. If $0 < x < 1$, then $x^n < 1$ for $n \geq 2$. If $k \geq 1$, we then get $(x^n)^k \leq x^n$ as above. In particular, we get $(x^n)^{n-1} \leq x^n$ and we note that $x^{n^2-n} = (x^n)^{n-1}$. Hence

$$\sum_{n=1}^{\infty} x^{n^2-n} = 1 + \sum_{n=2}^{\infty} x^{n^2-n} \leq 1 + \sum_{n=2}^{\infty} x^n$$

Since the latter series converges, it follows that the first series converges as well.

(1 mark) (f) $\sum_{n=0}^{\infty} \frac{x^n}{n^4}$ for $x > 1$.

**Solution:** Since $x > 1$, we have seen that $x^n > n^4$ for sufficently large $n$. Let $n_0$ be such that $x^n > n^4$ for $n > n_0$. It follows that

$$\sum_{n=0}^{\infty} \frac{x^n}{n^4} > \sum_{n=0}^{n_0} \frac{x^n}{n^4} + \sum_{n=n_0+1}^{\infty} 1$$

Since the second term on the right-hand side diverges to infinity, so does the left-hand side.

(1 (bonus)) (g) $\sum_{n=1}^{\infty} n \cdot x^n$ for $0 < x < 1$.

2. Which of the following sequences is *eventually increasing* and is *bounded above*?

(1 mark) (a) $1 - \frac{n+1}{2n^2-n}$

**Solution:** We first see that

$$\begin{aligned}(n+1)(2(n+1)^2 - (n+1)) &= (n^2 + 2n + 1)(2n - 1) \\ &= 2n^3 + 3n^2 - 1 \\ &> 2n^3 + 3n^2 - 2n \\ &= (n+2)(2n^2 - n)\end{aligned}$$

This shows that the sequence $\frac{n+1}{2n^2-n}$ is decreasing. Hence, $1 - \frac{n+1}{2n^2-n}$ is increasing. Next, we note that $\frac{n+1}{2n^2-n} > 0$ for $n \geq 2$. Hence, this sequence is bounded above.

(1 mark) (b) $\left(1+\frac{n}{n^2+1}\right)^n$

**Solution:** We note that

$$\left(1+\frac{n}{n^2+1}\right)^n = \sum_{k=0}^{n}\left(\frac{n^k}{(n^2+1)^k}\cdot\frac{n(n-1)\cdots(n-k+1)}{k!}\right)$$

We note that, for $0 \le a \le n$

$$(n-a)\cdot\frac{n}{n^2+1} = \left(1-\frac{a}{n}\right)\cdot\frac{1}{1+\frac{1}{n^2}}$$

Now, for $m > n$, we have

$$\begin{aligned}\frac{a}{n} &> \frac{a}{m}\\ 1-\frac{a}{n} &< 1-\frac{a}{n}\\ 1+\frac{1}{n^2} &> 1+\frac{1}{m^2}\\ \frac{1}{1+\frac{1}{n^2}} &< \frac{1}{1+\frac{1}{m^2}}\end{aligned}$$

It follows that

$$\frac{n^k}{(n^2+1)^k}\cdot\frac{n(n-1)\cdots(n-k+1)}{k!} < \frac{m^k}{(m^2+1)^k}\cdot\frac{m(m-1)\cdots(m-k+1)}{k!}$$

Thus the sequence is increasing. We also note that $n/(n^2+1) < 1/n$, so

$$\left(1+\frac{n}{n^2+1}\right)^n < \left(1+\frac{1}{n}\right)^n$$

Since the latter sequence is bounded, so is the first sequence.

(1 mark) (c) The sequence with $x_1 = 1$ and

$$x_{n+1} = \frac{2x_n+1}{x_n+1}$$

**Solution:** We note that $x_2 = 3/2 > x_1$. Now, assume that we are given

$x_{n+1} > x_n$. This means that

$$\begin{aligned} \frac{2x_n+1}{x_n+1} &> x_n \\ 2x_n+1 &> x_n^2 + x_n \\ x_n + 1 &> x_n^2 \end{aligned}$$

We then check

$$\begin{aligned} x_{n+1} + 1 - x_{n+1}^2 &= \frac{2x_n+1}{x_n+1} + 1 - \left(\frac{2x_n+1}{x_n+1}\right)^2 \\ &= \frac{(3x_n+2)(x_n+1) - (x_n+1)^2}{(x_n+1)^2} \\ &= \frac{x_n + 1 - x_n^2}{(x_n+1)^2} > 0 \end{aligned}$$

It follows that

$$x_{n+2} = \frac{2x_{n+1}+1}{x_{n+1}+1} > x_{n+1}$$

Thus, by the principle of induction, the sequence is increasing. It follows that we have $x_n \geq 1$ for all $n$. Hence,

$$x_{n+1} = \frac{2x_n+1}{x_n+1} = 2 - \frac{1}{x_n+1} < 2$$

for all $n$. So the sequence is bounded.

(1 mark) (d) The sequence with $x_1 = 1$ and

$$x_{n+1} = \frac{2x_n+3}{x_n+2}$$

**Solution:** We check that $x_2 = 5/3 > x_1$. Let us assume that $x_{n+1} > x_n$ for some $n$. This means

$$\begin{aligned} \frac{2x_n+3}{x_n+2} &> x_n \\ 2x_n+3 &> x_n^2 + 2x_n \\ 3 &> x_n^2 \end{aligned}$$

We then check

$$
\begin{aligned}
3 - x_{n+1}^2 = 3 - \left(\frac{2x_n + 3}{x_n + 2}\right)^2 \\
&= \frac{3(x_n + 2)^2 - (2x_n + 3)^2}{(x_n + 2)^2} \\
&= \frac{3x_n^2 + 12x_n + 12 - (4x_n^2 + 12x_n + 9)}{(x_n + 2)^2} \\
&= \frac{3 - x_n^2}{(x_n + 2)^2} > 0
\end{aligned}
$$

It follows that

$$
\begin{aligned}
x_{n+2} - x_{n+1} = \frac{2x_{n+1} + 3}{x_{n+1} + 2} - x_{n+1} \\
&= \frac{(2x_{n+1} + 3 - (x_{n+1}^2 + 2x_{n+1})}{x_{n+1} + 2} \\
&= \frac{3 - x_{n+1}^2}{(x_{n+1} + 2)} > 0 = \frac{3 - x_n^2}{(x_n + 2)^2} > 0
\end{aligned}
$$

By the principle of induction, we see that $x_n$ is an increasing sequence and it is bounded above by 2.