Limits

So far, we have studied increasing sequences (or, with a minus sign, decreasing sequences). What about sequences that are *not* monotonic? Such sequences arise for measurement reasons as well as mathematical reasons.

A measurement does not *always* approach the "exact" value from one side. As someone who has done experiments will testify, sometimes one overshoots the target and at other times one goes below.

Mathematically, we note that if $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are increasing sequences of positive numbers then $(x_n - y_n)_{n\geq 1}$ and $(x_n/y_n)_{n\geq 1}$ need not be increasing *or* decreasing. However, in both cases, we would like to calculate what happens "as n goes to infinity".

In this section we will study some properties of such, more general, sequences.

Supremum and Infimum

Given a bounded (above and below) sequence $(x_n)_{n\geq 1}$, we can define a new sequence by defining

$$y_n = \max\{x_1, \ldots, x_n\}$$

We can always find the maximum of a finite set of numbers by pair-wise comparison. Indeed, we note that $y_1 = x_1$. Having found y_n, we put $y_{n+1} = \max\{y_n, x_{n+1}\}$ which is just a pair-wise comparison.Now, by the last identity it is clear that $(y_n)_{n\geq 1}$ is a bounded non-decreasing sequence. So, it has a least upper bound y. We define the *supremum* of $(x_n)_{n\geq 1}$ to be y.

First of all, we note that $y \geq y_n \geq x_n$ for all n. Next, if $z < y$, then, since y is the least upper bound of $(y_n)_{n\geq 1}$, there is a p for which $y_p > z$. Since $y_p = \max\{x_1, \ldots, x_p\}$, we see that $y_p = x_q$ for *some* q between 1 and p. It follows that $z > x_q$. In other words, we have also shown that y is the smallest number which is an upper bound for $(x_n)_{n\geq 1}$. We just use a different word "supremum" and notation $\sup(x_n)_{n\geq 1}$ to make it clear that we are *not* assuming that x_n is increasing.

As a word of warning, we note that if we have a *finite* collection of numbers $\{x_1, \ldots, x_n\}$, then the maximum of these *is* one of the x_q's. However, for an infinite sequence $(x_n)_{n\geq 1}$, there is no reason why $\sup(x_n)_{n\geq 1}$ should be one of these numbers. In fact, it is very often *not* one of them! This is why we use a different word from "maximum".

We similarly definite the *infimum* $\inf(x_n)_{n\geq 1}$ to be $-\sup(-x_n)_{n\geq 1}$. We note that if $w_n = \max\{-x_1, \ldots, x_n\}$, then $-w_n = \min\{x_1, \ldots, x_n\}$. It follows that $-w_n$ is decreasing and its greatest lower bound is the negative of the least upper bound of w_n. So, we see that $\inf(x_n)_{n\geq 1} \leq x_n$ and is the largest number with this property.

We note that these two definitions *extend* the notions of least upper bound of an increasing sequence and greatest lower bound of a decreasing sequence to more general sequences.

Limit supremum, limit infimum and limit

Given a bounded sequence $(x_n)_{n\geq 1}$, consider the sequence $(x_{n+1})_{n\geq 1}$. We note that if

$$z_n = \max\{x_2, \ldots, x_{n+1}\}$$

then $z_n \leq y_{n+1}$, where $y_{n+1} = \max\{x_1, \ldots, x_{n+1}\}$. It follows that the least upper bound of $(z_n)_{n\geq 1}$ is less than or equal to the least upper bound of $(y_n)_{n\geq 1}$. In terms of the definitions given in the previous section we see that

$$\sup(x_{n+1})_{n\geq 1} \leq \sup(x_n)_{n\geq 1}$$

Equivalently, we note that

$$\sup(x_n)_{n\geq 2} \leq \sup(x_n)_{n\geq 1}$$

If we use the notation $s_k = \sup(x_n)_{n\geq k}$, then this shows that $(s_k)_{k\geq 1}$ is a (bounded) *non-increasing* sequence. Hence, it has a greatest lower bound $s = \inf(s_k)_{k\geq 1}$. We define

$$\limsup(x_n)_{n\geq 1} = \inf(\sup(x_n)_{n\geq k})_{k\geq 1}$$

and call it the *limit supremum* (or limit superior) of x_n. Similarly,

$$\liminf(x_n)_{n\geq 1} = \sup(\inf(x_n)_{n\geq k})_{k\geq 1}$$

Finally, we say that a sequence $(x_n)_{n\geq 1}$ has a limit x if

$$\limsup(x_n)_{n\geq 1} = \liminf(x_n)_{n\geq 1} = x$$

In this case, we use the notation $\lim(x_n)_{n\geq 1} = x$ and say that the sequence $(x_n)_{n\geq 1}$ has *limit* x.

Simplest example

Given a sequence $(x_n)_{n\geq 1}$ and a number x with the property that, for all n we have $x - 1/n \leq x_n \leq x + 1/n$.

First of all we note that

$$x = \sup(x - 1/n)_{n\geq k} \leq \sup(x_n)_{n\geq k} \leq \sup(x + 1/n)_{n\geq k} = x + 1/k$$

Hence,

$$x \leq \inf(\sup(x_n)_{n\geq k})_{k\geq 1} \leq \inf(x + 1/k)_{k\geq 1} = x$$

This shows that $\limsup(x_n)_{n\geq 1} = x$. We similarly show that $\liminf(x_n)_{n\geq 1} = x$.

In other words, given a sequence $(x_n)_{n\geq 1}$ with the property that for all n, we have $|x_n - x| \leq 1/n$, the limit of the sequence is x.

Subsequences

Given a sequence $(x_n)_{n\geq 1}$ and a sequence of natural numbers

$$n_1 < n_2 < n_3 < \cdots$$

we can form a new sequence by defining $y_p = x_{n_p}$. This sequence $(y_p)_{p\geq 1}$ is called a *subsequence* of the sequence $(x_n)_{n\geq 1}$.

Since n_1 is a natural number, we have $n_1 \geq 1$. Given that $n_p \geq p$, we note that n_{p+1} is a natural number greater than n_p. So $n_{p+1} \geq n_p + 1 \geq p + 1$. By the principle of induction, we see that $n_p \geq p$.

As a consequence we see

$$\sup(y_p)_{p\geq k} = \sup(x_{n_p})_{p\geq k} \leq \sup(x_n)_{n\geq k}$$

It follows that

$$\limsup(y_p)_{p\geq 1} = \inf(\sup(y_p)_{p\geq k})_{k\geq 1} \leq \inf(\sup(x_n)_{n\geq k})_{k\geq 1} = \limsup(x_n)_{n\geq 1}$$

In words, the limit superior of a sequence is greater than or equal to the limit superior of a subsequence.

Similarly, we show that

$$\liminf(y_p)_{p\geq 1} \geq \liminf(x_n)_{n\geq 1}$$

In words, the limit inferior of a sequence is less than or equal to the limit inferior of a subsequence.

Now, suppose that $x = \limsup(x_n)_{n\geq 1}$. By definition, if $s_p = \sup(x_n)_{n\geq p}$, then $x = \inf(s_p)_{p\geq 1}$. Since x is the greatest lower bound of s_p (which is decreasing sequence), it follows that for each k, there is a p_k so that $s_p \leq x + 1/k$ for all $p \geq p_k$. By the definition of s_p, there is an $n_k \geq p$ so that $x_{n_k} \geq s_p - 1/k$. Now, $x \leq s_p$ so $x - 1/k \leq s_p - 1/k \leq x_{n_k}$. Moreover, $x_{n_k} \leq s_p \leq x + 1/k$. So we see that $|x_{n_k} - x| \leq 1/k$. Furthermore, we can choose $p \geq p_k$ and $p > n_{k-1}$, so we can ensure n_k is an increasing sequence of natural numbers.

In other words, we have found an increasing sequence $(n_k)_{k\geq 1}$ of natural numbers so that the subsequence $(x_{n_k})_{k\geq 1}$ of x_n has the property $|x_{n_k} - x| \leq 1/k$. As seen above, this means that this subsequence has limit x.

> If $x = \limsup(x_n)_{n\geq 1}$, then there is a subsequence $(x_{n_k})_{k\geq 1}$ which has limit x.

Similarly, we can prove

> If $x = \liminf(x_n)_{n\geq 1}$, then there is a subsequence $(x_{n_k})_{k\geq 1}$ which limit x.

One can state the results of this subsection in a consolidated fashion as follows.

> The limit superior of a sequence is the largest number that is the limit of a subsequence.

Similarly,

> The limit inferior of a sequence is the smallest number that is the limit of a subsequence.

Convergence property

Given that a sequence $(x_n)_{n\geq 1}$ has a limit and that limit is x. If we define

$$s_p = \sup(x_n)_{n\geq p}$$
$$r_q = \inf(x_n)_{n\geq q}$$

Then, x is the greatest lower bound of $(s_p)_{p\geq 1}$ and is also the least upper bound of $(r_q)_{q\geq 1}$. Given a natural number k, this means that there is a p_k so that $s_p \leq x + 1/k$ for all $p \geq p_k$. Similarly, there is a q_k so that $r_q \geq x - 1/k$ for all $q \geq q_k$. So, if $n \geq n_k = \max\{p_k, q_k\}$, then $s_n \leq x + 1/k$ and $r_n \geq x - 1/k$.

Now, by definition of s_n, we have $x_n \leq s_n$. So $x_n \leq x+1/k$ for $n \geq n_k$. Similarly, by definition of r_n, we have $x_n \geq r_n$. So $x_n \geq x - 1/k$ for $n \geq n_k$.

In other words, we have shown that, if the sequence $(x_n)_{n\geq 1}$ has a limit x, then for all k, there is an n_k so that $|x_n - x| \leq 1/k$ for $n \geq n_k$. So, *all but a finite number of terms* of the sequence are within $1/k$ of the limit of the sequence. This allows us to think of the limit in terms of distance. So we also use the phrase $(x_n)_{n\geq 1}$ *converges* to its limit x.

Note that, we have already proved the converse of this statement above as the simplest example of a sequence with a limit.

Cauchy Criterion

We first want to show that if two convergent sequences are "eventually" *arbitrarily* close to each other, then they have the same limit. More precisely, given two convergent sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ that converge to x and y respectively. Further assume that, for every k, there is an n_k so that $|x_n - y_n| \leq 1/k$ for $n \geq n_k$. We want to show that $x = y$.

We note that $x_n \leq y_n + 1/k$ for all $n \geq n_k$. This means that

$$\sup(x_n)_{n\geq p} \leq \sup(y_n)_{n\geq p} + 1/k \text{ for } p \geq n_k$$

It follows that for *all* k, we have

$$x = \limsup(x_n)_{n\geq p} \leq y = \limsup(y_n)_{n\geq p} + 1/k$$

This means that $x \leq y$. By interchanging the roles of (x_n) and (y_n) we see that $y \leq x$. Thus, we have $x = y$.

Now suppose that $(x_n)_{n\geq 1}$ is a sequence satisfying (the *Cauchy property*),

> for every natural number k, there is an n_k so that $|x_p - x_q| \leq 1/k$ for $p, q \geq n_k$.

In other words, all but finitely many terms of the sequence are (pairwise) within $1/k$ of each other. We claim that the sequence has a limit. Let $r = \limsup(x_n)_{n\geq 1}$ and $s = \liminf(x_n)_{n\geq 1}$. We want to prove that $r = s$.

As seen above, there is a subsequence $(y_k) = (x_{p_k})_{k\geq 1}$ for which r is the limit and there is a subsequence $(z_k) = (x_{q_k})_{k\geq 1}$ for which s is the limit. Given a natural number k, from the Cauchy property, we see that if we choose $m \geq n_k$, then $p_m \geq m \geq n_k$ and $q_m \geq m \geq n_k$. It follows that $|y_m - z_m| \leq 1/k$ for all such m. By what has been proved above, we see that $r = s$ is the limit of the sequence $(x_n)_{n\geq 1}$.

Subtraction and division

Subtraction is just addition of the negative of a number and division is just multiplication of the reciprocal. So, what we need is to prove the arithmetic properties of the limit.

Given two convergent sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ that converge to x and y respectively. We want to show that $(x_n + y_n)_{n\geq 1}$ converges to $x + y$. As seen above, for each k, there is a p_k so that $|x_p - x| \leq 1/(2k)$ for $p \geq p_k$ and there is a q_k so that $|y_p - y| \leq 1/(2k)$ for $q \geq q_k$. So, if $n \geq n_k = \max\{p_k, q_k\}$, we have

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| \leq 1/k \text{ for } n \geq n_k$$

As seen above, this means that $x_n + y_n$ converges to $x + y$.

In this proof we have used the *triangle inequality*

$$|a + b| \leq |a| + |b|$$

which is an important ingredient in many proofs in analysis.

Given two convergent sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ that converge to x and y respectively. We want to show that $(x_n \cdot y_n)_{n\geq 1}$ converges to $x \cdot y$. As above, for each k, we want to find n_k such that $|(x_n \cdot y_n) - (x \cdot y)| \leq 1/k$ for $n \geq n_k$.

Since $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ have limits x and y, the collection of all these numbers is bounded. By the Archimedean principle, there is a natural number M so that $|x_n| \leq M$ for all n and $|y_n| \leq M$ for all n; we also assume that $|x| \leq M$ and $|y| \leq M$.

Since $(x_n)_{n\geq 1}$ converges to x, there is a p_k so that $|x_n - x| \leq 1/(2kM)$ for all $n \geq p_k$. Similarly, there is a q_k so that $|y_n - y| \leq 1/(2kM)$ for all $n \geq q_k$. It follows that, if $n \geq n_k \max\{p_k, q_k\}$, then

$$|x_n \cdot y_n - x \cdot y| \leq |(x_n - x)||y_n| + |x||(y_n - y)| \leq 1/k$$

as required.

To complete the argument for subtraction, we note that if $(x_n)_{n\geq 1}$ has limit x, then

$$-x = -\limsup(x_n)_{n\geq 1} = \liminf(-x_n)_{n\geq 1}$$

and

$$-x = -\liminf(x_n)_{n\geq 1} = \limsup(-x_n)_{n\geq 1}$$

Hence, $(-x_n)_{n\geq 1}$ has limit $-x$.

To complete the argument for division, we need to examine the sequence $(1/x_n)_{n\geq 1}$ when we are given a sequence $(x_n)_{n\geq 1}$ with limit x. (Note that some terms of this sequence may not even be meaningful since x_n may be 0 for some n!)

First, we need to put the condition $x \neq 0$, since division by 0 is not permissible. Now, by using the previous paragraph, we can assume that $x > 0$, replacing x by $-x$ if necessary. By the Archimedean principle, there is a natural number M so that $x \geq 2/M$. Since $(x_n)_{n\geq 1}$ has limit x, there is a n_0 so that $|x_n - x| \leq 1/M$ for all $n \geq n_0$. It follows that $x_n \geq x - 1/M \geq 1/M$ for $n \geq n_0$. In particular, this means that $1/x_n$ *is* meaningful for $n \geq n_0$. Now, for each natural number k, there is an $n_k \geq n_0$ such that $|x_n - x| \leq 1/(k \cdot M^2)$. It follows that

$$|(1/x_n) - (1/x)| = \frac{|x_n - x|}{x_n \cdot x} \leq 1/k$$

This proves that the sequence $(1/x_n)_{n\geq n_0}$ converges to $1/x$.

Some examples

Let us examine some examples where this concept of limits, which extends the least upper bound and greatest lower bound, clarifies some properties of numbers.

Negative interest

In the study of compound interest we assumed that the rate of interest is positive. This is a reasonable assumption for *that* application. However, there *are* applications, like the calculation of the mass of rocket fuel, where it is worth understanding what happens to the sequence $((1 + r/n)^n)_{n\geq 1}$ for $r < 0$. Equivalently, we can study the sequence $((1 - r/n)^n)_{n\geq 1}$ for $r > 0$.

If we apply the Binomial theorem to this as we did earlier we get the expression$

$$\left(1 - \frac{r}{n}\right)^n = 1 - \binom{n}{1}\frac{r}{n} + \binom{n}{2}\frac{r^2}{n^2} + \cdots (-1)^n \binom{n}{n}\frac{r^n}{n^n}$$

The signs *alternate* making it difficult to see whether this is increasing or decreasing. On the other hand, we note that (for $n \geq 2$)

$$\left(1 - \frac{1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n = \frac{1}{\left(1 + \frac{1}{n-1}\right)^n}$$

In other words, we see that

$$\left(1 - \frac{1}{n+1}\right)^{n+1} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{\left(1+\frac{1}{n}\right)}$$

Since the sequence $((1 + 1/n)^n)_{n\geq 1}$ is increasing and bounded it has a limit; call it e. The sequence $1 + 1/n$ is decreasing at its greatest lower bound is 1. By the arithmetic arguments made above we see that

$$\lim \left(\left(1 - \frac{1}{n+1}\right)^{n+1}\right)_{n\geq 1} \lim \left(\frac{1}{\left(1+\frac{1}{n}\right)^n}\right)_{n\geq 1} \cdot \lim \left(\frac{1}{\left(1+\frac{1}{n}\right)}\right)_{n\geq 1} = \frac{1}{e} \cdot \frac{1}{1} = 1/e$$

Square roots

Given a number $d > 1$ (say 5) we picked a number f such that $f^2 > d$ (in our example say 3). We then define a sequence $x_1 = 1$ and

$$x_{n+1} = \frac{fx_n + d}{x_n + f}$$

We proved (by induction) that $(x_n)_{n\neq 1}$ is an increasing and bounded sequence. The claim that we did *not* prove is that, if x is the least upper bound, then $x^2 = d$. We can now prove this.

Since x is the least upper bound of the increasing sequence $(x_n)_{n\geq 1}$, we see that $x = \lim(x_n)_{n\geq 1}$. We then note, by the arithmetic properties proved above that

$$\lim(x_{n+1})_{n\geq 1} = \lim \left(\frac{fx_n + d}{x_n + f}\right)_{n\geq 1} = \frac{fx + d}{x + f}$$

For any sequence $\lim(x_{n+k})_{n\geq 1} = \lim(x_n)_{n\geq 1}$ since the first few terms clearly do not matter! We thus obtain the identity

$$x = \frac{fx + d}{x + f}$$

This gives us $x^2 = d$!