Solutions to Assignment 2

1. Compare the following sequences to decide which one is *eventually* larger.

(1 mark) (a) The sequence with general term 10^{15}

$$10^{15}, 10^{15}, 10^{15}, \ldots$$

versus the sequence with general term $\frac{n}{10^{15}}$

$$\frac{1}{10^{15}}, \frac{2}{10^{15}}, \frac{3}{10^{15}}, \ldots$$

> **Solution:** For $n > 10^{30}$ we have $\frac{n}{10^{15}} > 10^{15}$.

(1 mark) (b) The sequence with general term $n \cdot 10^{10}$

$$1 \cdot 10^{10}, 2 \cdot 10^{10}, 3 \cdot 10^{10}, \ldots$$

versus the sequence with general term $n^2/10^{15}$

$$\frac{1^2}{10^{15}}, \frac{2^2}{10^{15}}, \frac{3^2}{10^{15}}, \ldots$$

> **Solution:** For $n > 10^{25}$ we have $\frac{n^2}{10^{15}} > n \cdot 10^{10}$.

(1 mark) (c) The sequence with general term $(n + 1000) \cdot 10^{10}$

$$(1001) \cdot 10^{10}, (1002) \cdot 10^{10}, (1003) \cdot 10^{10}, \ldots$$

versus the sequence with general term $n^2/10^{15}$

$$\frac{1^2}{10^{15}}, \frac{2^2}{10^{15}}, \frac{3^2}{10^{15}}, \ldots$$

> **Solution:** For $n > 10^{28}$ we have $\frac{n^2}{10^{15}} > n \cdot 10^{13}$. For $n > 1$ we have $n \cdot 10^3 > (n + 1000)$, hence $n \cdot 10^{13} > (n + 1000) \cdot 10^{10}$.

(1 mark) (d) The sequence with general term $n \cdot 10^{10}$

$$1 \cdot 10^{10}, 2 \cdot 10^{10}, 3 \cdot 10^{10}, \ldots$$

versus the sequence with general term $(n^2 - n)/10^{15}$

$$0 = \frac{0}{10^{15}}, \frac{2}{10^{15}}, \frac{6}{10^{15}}, \ldots$$

Solution: For $n > 10^{25}+1$, we have $n^2 > (10^{25}+1)\cdot n$, and so, $(n^2-n) > n\cdot 10^{25}$. It follows that $(n^2-n)/10^{15} > n \cdot 10^{10}$ for such n.

(1 (bonus)) (e) The sequence with general term 2^n

$$2^1, 2^2, 2^3, \ldots$$

versus the Fibonacci sequence with general term $F(n) = F(n-1)+F(n-2)$ starting with $F(1) = 5$ and $F(2) = 8$.

$$5, 8, 13, \ldots$$

2. Give the properties of each sequence out of:

eventually increasing, eventually decreasing, neither, bounded, unbounded

(1 mark) (a) The sequence with general term $n/(n+1)$.

$$1/2, 2/3, 3/4, \ldots$$

Solution: We see that $n/(n+1) = 1 - 1/(n+1)$. Since $1/(n+1)$ *decreases* with increasing n, the given sequence is increasing. Moreover, we see that 1 is an upper bound. Since all the fractions are positive 0 is a lower bound.

(1 mark) (b) The sequence with general term $5(n+1)^2/(n^3+2n)$

$$20/3, 45/12, 80/33, \ldots$$

Solution: To compare successive terms of this sequence we write

$$\begin{aligned}
&5(n+1)^2((n+1)^3+2(n+1)) - 5((n+1)+1)^2(n^3+2n)\\
&= 5\left(\left(n^2+2n+1\right)\left(n^3+3n^2+5n+3\right) - \left(n^2+4n+4\right)\left(n^3+2n\right)\right)\\
&= 5\left(\left(n^5+5n^4+12n^3+16n^2+11n+3\right) - \left(n^5+4n^4+6n^3+8n^2+8n\right)\right)\\
&= 5(n^3+3n^2+5n+3)
\end{aligned}$$

This number is positive for all positive values of n. Hence, the sequence is *decreasing*. It follows that $10/3$ is an upper bound! It is also clear that all the fractions are positive, so 0 is a lower bound.

(1 mark) (c) The sequence with general term $n^{100}/2^n$

$$1/2, 2^{98}, 3^{100}/8, \ldots$$

Solution: To compare successive terms of this sequence we write

$$\frac{n^{100}2^{n+1}}{(n+1)^{100}2^n} = 2\left(1 - \frac{1}{n+1}\right)^{100}$$

Now, $(1 - 1/(n+1))^{100} > 1/2$ for $n > 200$, so this fraction is *greater* than 1. It follows that the sequence is *eventually decreasing*. Moreover, it is bounded above the *maximum* of the first 200 terms! Since all the fractions are positive, so 0 is a lower bound.

(1 mark) (d) The sequence with general term $2^{2n-1}/(10^n)$

$$2/10, 8/100, 32/1000, \ldots$$

Solution: To compare successive terms of this sequence we write

$$\frac{2^{2n-1}10^{n+1}}{2^{2n+1}10^n} = \frac{10}{4} > 1$$

It follows that the sequence is decreasing. Since $10/4 > 2$ it follows that the terms of the sequence are less than $(2/10)/2^{n-1}$. Since this sequence is bounded above by $(2/10)$, our original sequence is also bounded above by $2/10$.

(1 (bonus)) (e) The sequence with general term $n \cdot \sin(1/n)$

$$\sin(1), 2\sin(1/2), 3\sin(1/3), \ldots$$

3. Which of the following sequences has an *upper* bound and which does not.

(1 mark) (a) The sequence with general term $n^2 - 100 \cdot n$.

Solution: We note that $n^2 - 100 \cdot n > n$ for $n > 101$. It follows that the sequence is not bounded above.

(1 mark) (b) The sequence with general term $1000 \cdot n^2 - 2^n$.

Solution: We note that $2^{10} = 1024 > 1000$. Now, $2^{n-10} > n^2$ if $n = 20$. Moreover,

$$\frac{2^{(n+1)-10}}{2^{n-10}} = 2 > \frac{n+1}{n}^2$$

For $n > 3$. It follows that $2^{n-10} > n^2$ for *all* $n > 20$. Thus the terms of the sequence are *negative* for $n > 20$. The maximum of the first 20 terms is therefore an upper bound.

(1 (bonus)) (c) The sequence with general term $1 + 1/2 + \cdots + 1/n!$ where $n! = 1 \cdot 2 \cdots n$ is the factorial of n.