Differential Equations for Scientists
IDC205 End-Sem Exam 29 October 2018

Read all the questions **carefully** before writing anything.
Write your calculations **neatly**.
There will be **no** partial credit for marginal scribbles!
Answer each part of each question on a **separate** page.
Each part of Question 1 can be done within 1 page.
You have 3 hours to complete this exam.

1. Find the mathematical expression asked for in each case. These are chosen so that *very little* calculation is required. Do not write long answers or marks will be deducted!

   1 Mark is for a correct answer and 2 marks for a clear calculation of this answer in each case.

(3 marks) (a) The exponential matrix of the matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution: This matrix is nilpotent (1 Mark). So its exponential is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 7/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

(1 Mark for *three* terms. 1 Mark for answer.)

(3 marks) (b) The Fourier series of the function $f(\theta) = \cos\theta(\cos\theta + \sin\theta)$ for θ in $[0, 2\pi]$. (Series in terms of $\cos n\theta$ and $\sin n\theta$.)

Solution: We have $\cos 2\theta = 2\cos^2\theta - 1$, and $\sin 2\theta = 2\sin\theta\cos\theta$. (1 Mark) So

$$\cos\theta(\cos\theta+\sin\theta) = \cos^2\theta+\cos\theta\sin\theta = \frac{\cos 2\theta + 1}{2}+\frac{\sin 2\theta}{2} = \frac{1}{2}+\frac{\cos 2\theta}{2}+\frac{\sin 2\theta}{2}$$

(1 Mark for this calculation and 1 Mark for answer.)

(3 marks) (c) A value of the constant a for which there is a function $f(x, y)$ so that

$$\frac{\partial f}{\partial x} = x^6 y^{22} \text{ and } \frac{\partial f}{\partial y} = a x^7 y^{21}$$

Solution: We have (1 Mark for exact-ness check)

$$\frac{\partial}{\partial y}(x^6y^{22}) - \frac{\partial}{\partial x}(ax^7y^{21}) = (22 - 7a)x^6y^{21}$$

(1 Mark for the computation. 1 Mark for the answer.) So $a = 22/7$. (Which is *not* π!)

(3 marks) (d) A function $x(t)$ so that

$$\frac{d^2x}{dt^2} - 4x = 0 \text{ and } x(0) = 2 \text{ and } \frac{dx}{dt}(0) = 0$$

Solution: The general solution has the form $A\exp(2t) + B\exp(-2t)$. (1 Mark for this general solution). The conditions give the equations $A + B = 2$ and $2A - 2B = 0$. (1 Mark for using the conditions to get these linear equations.) In other words, $A = B = 1$. So the solution is $x(t) = \exp(2t) + \exp(-2t)$. (1 Mark for the answer)

(3 marks) (e) A function $x(t)$ so that

$$\frac{dx}{dt} = t^2x + \exp(t^3/3) \text{ and } x(0) = 0$$

Solution: The homogeneous equation

$$\frac{dx}{dt} = t^2x$$

Has the solution $x(t) = c\exp(t^3/3)$. (1 Mark for the solution of homogeneous equation.) By the method of variation of parameters we put $x(t) = c(t)\exp(t^3/3)$ as the proposed solution. (1 Mark for this) Substituting this in the inhomogeneous equation gives the equation

$$t^2c(t)\exp(t^3/3) + \frac{dc}{dt}\exp(t^3/3) = t^2c(t)\exp(t^3/3) + \exp(t^3/3)$$

This gives the equation $dc/dt = 1$ or $c(t) = t + a$ for a constant a. Using $x(0) = 0$, we get $c(t) = t$. Hence, the solution is $x(t) = t\exp(t^3/3)$. (1 Mark for the solution)

(3 marks) (f) The first 4 terms of the power series solution of

$$\frac{dx}{dt} = x^5 \text{ with } x(0) = 1$$

Solution: We differentiate iteratively and substitute using the equation to get (1 Mark for 2nd and 3rd identity.)

$$\frac{dx}{dt} = x^5$$
$$\frac{d^2x}{dt^2} = 5x^9$$
$$\frac{d^3x}{dt^3} = 45x^{13}$$

Now, using the Taylor series and $x(0) = 1$ we calculate the other derivatives at $t = 0$ to get $x(t) = 1 + t + 5t^2/2 + 15t^3/2 + \ldots$. (1 Mark for final answer)

(3 marks) (g) The first 3 terms of the power series solution of

$$\frac{d^2x}{dt^2} = x\frac{dx}{dt} + x^2 \text{ with } x(0) = 2 \text{ and } \frac{dx}{dt}(0) = 3$$

Solution: We use the given values at $t = 0$ in the equation to get the value of the second derivative at $t = 0$ as 10 (1 Mark). Now, we use the Taylor series (1 Mark) to get $x(t) = 2 + 3t + 5t^2 + \ldots$. (1 Mark for final answer.)

(3 marks) (h) A solution of the equation

$$t\frac{dx}{dt} = (1/2)x \text{ for } t > 0 \text{ with } x(1) = -1$$

Solution: We know that the general solution for

$$t\frac{dx}{dt} = ax \text{ for } t > 0$$

has the form $x = ct^a$. Hence, our solution is $x(t) = ct^{1/2}$ for some constant c. (1 Mark)
Given $x(1) = -1$, we get $x(t) = -t^{1/2}$. (1 Mark for using the equation to get the constant. 1 Mark for the final answer.)

(3 marks) (i) A solution of the equation

$$t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - 2x = 0 \text{ for } t > 0 \text{ with } x(1) = 0 \text{ and } \frac{dx}{dt}(1) = 3$$

Solution: Writing $x(t) = \sum_m a_m x^m$ and equating coefficients of x^m we obtain

$$(m(m-1) + 2m - 2)\, a_m = 0$$

(1 Mark for this indicial equation.) It follows that $a_m = 0$ unless $m(m-1) + 2m - 2 = 0$. The left-hand side simplifies to $m^2 + m - 2$ which is $(m-1)(m+2)$ and has the roots $m = 1, -2$.

Hence, the general solution has the form $At + B/t^2$. The given conditions say that $A + B = 0$ and $A - 2B = 3$. So $A = 1$ and $B = -1$. (1 Mark for using the conditions correctly.) In other words, the solution is $x(t) = t - 1/t^2$. (1 Mark for the final answer.)

(3 marks) (j) A solution $u(t, x)$ of the Heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

with $u(t, 0) = 0 = u(t, \pi)$ and $u(0, x) = \sin(2x)$ for all x in $[0, \pi]$.

Solution: The Fourier series of $\sin(2x)$ is "itself". In other words, $a_n = 0$ for all n and $b_n = 0$ for $n \neq 2$ and $b_1 = 1$. (1 Mark for noticing this.)

By the method of separation of variables the general solution has the form (1 Mark for general form.)

$$u(x, t) = \sum_{n=1}^{\infty} b_n \exp(-a^2 n^2 t) \sin(nx)$$

Thus, we obtain the solution $u(x, t) = \exp(-4a^2 t) \sin(2x)$. (1 Mark for final answer.)

2. Answer the following questions about linear ordinary differential equations. You may use the solution of one part to solve another part.

(4 marks) (a) Find the 2×2 matrix $G(t)$ so that

$$\frac{d}{dt} G(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} G(t)$$

and $G(0)$ is the identity matrix.

Solution: The matrices

$$A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ and } A(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

commute (1 Mark) with each other for all s and t. Hence, the solution is (1 Mark)

$$G(t) = \exp\left(\int_0^t A(t)dt\right) = \exp\begin{pmatrix} t & t^2/2 \\ 0 & t \end{pmatrix}$$

Now

$$\begin{pmatrix} t & t^2/2 \\ 0 & t \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} 0 & t^2/2 \\ 0 & 0 \end{pmatrix}$$

where these two matrices also commute with each other. (1 Mark) So,

$$\exp\begin{pmatrix} t & t^2/2 \\ 0 & t \end{pmatrix} = \exp\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & t^2/2 \\ 0 & 0 \end{pmatrix}$$

This shows us

$$G(t) = \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(t) \end{pmatrix} \cdot \exp\begin{pmatrix} 1 & t^2/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \exp(t) & \exp(t)t^2/2 \\ 0 & \exp(t) \end{pmatrix}$$

(1 Mark for the answer). We can also use the connection between the solutions given below.

(4 marks) (b) What is the general solution of

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + tx \end{pmatrix}$$

Solution: We can use the connection between the solutions as explained below. The direct solution is as follows.

First of all we have $dx/dt = x$ so $x = A\exp(t)$ for some constant A (1 Mark). Thus, the second equation becomes

$$\frac{dy}{dt} = y + At\exp(t)$$

We solve this by variation of parameters (1 Mark). $y(t) = c(t)\exp(t)$. This gives the equation

$$\frac{dc}{dt}\exp(t) + c(t)\exp(t) = c(t)\exp(t) + At\exp(t)$$

This gives $c(t) = B + At^2/2$ for some constant B (1 Mark). So the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A\exp(t) \\ B\exp(t) + At^2/2\exp(t) \end{pmatrix}$$

(1 Mark)

(2 marks) (c) What is the connection between the above two solutions?

Solution: The general solution in the second case has the form (1 Mark)

$$\begin{pmatrix} A\exp(t) \\ B\exp(t) + At^2/2\exp(t) \end{pmatrix} = \begin{pmatrix} \exp(t) & 0 \\ \exp(t)t^2/2 & \exp(t) \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix}$$

We notice that the equation in the second case has the form (1 Mark)

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

So the two problems are the same after interchanging the variables.

3. Answer the following questions related with the Laplace equation. You may use the solution of one part to solve another part.

(4 marks) (a) Find the Fourier series of the function $f(\theta)$ on $[-\pi, \pi]$ given by

$$f(\theta) = |\sin(\theta)|$$

Solution: We note that $f(-\theta) = f(\theta)$ so that $b_n = 0$ for all n. (1 Mark) Moreover a_n are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos(n\theta)d\theta = \frac{2}{\pi}\int_0^{\pi} \sin(\theta)\cos(n\theta)d\theta$$

For $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin(\theta)d\theta = \left.\frac{-2\cos(\theta)}{\pi}\right|_0^{\pi} = \frac{4}{\pi}$$

Now (1 Mark)

$$\sin(\theta)\cos(n\theta) = \frac{\sin((n+1)\theta) - \sin((n-1)\theta)}{2}$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \frac{\sin(2\theta)}{2}d\theta = \frac{2}{\pi}\left.\frac{-\cos(2\theta)}{4}\right|_0^{\pi} = 0$$

For $n \geq 2$,

$$
\begin{aligned}
a_n = \frac{2}{\pi}\int_0^{\pi} \frac{\sin((1+n)\theta) - \sin((n-1)\theta)}{2} d\theta \\
= \frac{2}{\pi}\left(\frac{-\cos((1+n)\theta)}{2(n+1)} + \frac{\cos((n-1)\theta)}{2(n-1)}\right)\Bigg|_0^{\pi} \\
= \begin{cases} 0 & n = 2k+1 \\ \dfrac{-4}{(n^2-1)\pi} & n = 2k \end{cases}
\end{aligned}
$$

(1 Mark for the integration and 1 Mark for the simplification.) We can also use the argument given in part 2 to directly reduce to the case $n = 2k$.
This gives

$$f(\theta) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k\theta)}{4k^2 - 1}$$

(1 Mark for this answer.)

(4 marks) (b) Find a function $u(x, y)$ for $x^2 + y^2 \leq 1$ that satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

where $u(x, y) = |x|$ for $x^2 + y^2 = 1$.

Solution: Using polar coordinates, we have derived the solution of the Laplace equation in the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$$

where a_n's and b_n are the Fourier coefficients of $u(1, \theta) = |\cos\theta||$.
Put $g(\theta) = |\cos\theta|$. We note that $g(-\theta) = g(\theta)$ so that $b_n = 0$ for all n. (1 Mark) Moreover a_n are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi} g(\theta)\cos(n\theta) d\theta$$

Further, we have $g(\theta) = g(\pi - theta)$ so $a_n = 0$ unless $n = 2k$ and in that case

$$a_{2k} = \frac{4}{\pi}\int_0^{\pi/2} g(\theta)\cos(n\theta) d\theta = \frac{4}{\pi}\int_0^{\pi/2} \cos(\theta)\cos(n\theta) d\theta$$

For $n = 0$,

$$a_0 = \frac{4}{\pi}\int_0^{\pi/2} \cos(\theta) d\theta = \frac{4\sin(\theta)}{\pi}\Bigg|_0^{\pi/2} = \frac{4}{\pi}$$

Now (1 Mark)

$$\cos(\theta)\cos(n\theta) = \frac{\cos((n+1)\theta) + \cos((n-1)\theta)}{2}$$

For $k \geq 1$,

$$\begin{aligned} a_{2k} = \frac{4}{\pi}\int_0^{\pi/2} \frac{\cos((2k+1)\theta) + \cos((2k-1)\theta)}{2} d\theta \\ = \frac{4}{\pi}\left(\frac{\sin((2k+1)\theta)}{2(2k+1)} + \frac{\sin((2k-1)\theta)}{2(2k-1)}\right)\Bigg|_0^{\pi/2} \\ = \frac{(-1)^k \cdot 4}{(4k^2-1)\pi} \end{aligned}$$

(1 Mark for the integration and 1 Mark for the simplification.) This gives

$$u(r,\theta) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty} r^{2k}\frac{(-1)^k\cos(2k\theta)}{4k^2-1}$$

Using the Polynomials

$$\begin{aligned} P_n(x,y) &= \frac{(x+\iota y)^n + (x-\iota y)^n}{2} \\ Q_n(x,y) &= \frac{(x+\iota y)^n - (x-\iota y)^n}{2\iota} \end{aligned}$$

We can write this as

$$u(x,y) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1}P_{2k}(x,y)$$

(2 marks) (c) What is the connection between the above two solutions?

Solution: We note that

$$\sin(\theta + \pi/2) = \cos(\theta)$$

The interchange of x and y co-ordinates in the Laplace equation has the same result. (1 Mark)
Hence, $f(\theta + \pi/2) = g(\theta)$. So, the Fourier series

$$f(\theta) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k\theta)}{4k^2-1}$$

leads to (1 Mark)

$$g(\theta) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k\theta + k\pi)}{4k^2 - 1} = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k \cos(2k\theta)}{4k^2 - 1}$$