

Solutions to Quiz 7

(5 marks) 1. Solve the Dirichlet problem in the unit disk:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $x^2 + y^2 \leq 1$ under the conditions

$$u(x, y) = x + y^2 \text{ for } x^2 + y^2 = 1$$

Solution: We convert to polar co-ordinates (r, θ) (1 Mark). The problem is then restated as:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

with the boundary condition

$$u(1, \theta) = \cos \theta + \sin^2 \theta$$

Writing this as a Fourier Series (2 Marks) we have

$$u(1, \theta) = \frac{1}{2} + \cos \theta - \frac{\cos(2\theta)}{2}$$

It follows that the solution is (1 Mark)

$$u(r, \theta) = \frac{1}{2} + r \cos \theta - r^2 \frac{\cos(2\theta)}{2}$$

Converting this in terms of x and y , we get (1 Mark)

$$u(x, y) = \frac{1}{2} + x - \frac{x^2 - y^2}{2}$$