Solutions to Assignment 7

1. Find the Fourier series for the following functions:

(a)

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ 0 & -\pi \leq x < 0 \end{cases}$$

Solution: We calculate

$$a_0 = \frac{1}{\pi}\int_0^\pi dx = 1$$

Next, we have (for $n \geq 1$)

$$a_n = \frac{1}{\pi}\int_0^\pi \cos(nx)dx = \left.\frac{\sin(nx)}{n\pi}\right|_{x=0}^{\pi} = 0$$

Similarly, we have

$$b_n = \frac{1}{\pi}\int_0^\pi \sin(nx)dx = \left.\frac{-\cos(nx)}{n\pi}\right|_{x=0}^{\pi} = \begin{cases} 0 & n = 2k \\ \frac{2}{n\pi} & n = 2k+1 \end{cases}$$

Putting it all together the Fourier series is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{(2k+1)}$$

(b)

$$f(x) = \begin{cases} \sin(x) & 0 \leq x \leq \pi \\ 0 & -\pi \leq x < 0 \end{cases}$$

Solution: We calculate

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin(x)dx = \left.\frac{-\cos(x)}{\pi}\right|_{x=0}^{\pi} = \frac{2}{\pi}$$

Next, we have

$$a_1 = \frac{1}{\pi}\int_0^\pi \sin(x)\cos(x)dx = \frac{1}{\pi}\int_0^\pi \frac{\sin(2x)}{2}dx = \frac{1}{\pi}\left.\left(\frac{-\cos(2x)}{4}\right)\right|_{x=0}^{\pi} = 0$$

Next, we have (for $n \geq 2$)

$$a_n = \frac{1}{\pi}\int_0^\pi \sin(x)\cos(nx)dx = \frac{1}{\pi}\int_0^\pi \frac{\sin((n+1)x) - \sin((n-1)x)}{2}dx$$
$$= \frac{1}{2\pi}\left(\frac{-\cos((n+1)x)}{n+1} + \frac{\cos((n-1)x)}{n-1}\right)\Bigg|_{x=0}^{\pi}$$
$$= \begin{cases} 0 & n = 2k+1 \\ \frac{-2}{(n^2-1)\pi} & n = 2k \end{cases}$$

Similarly, we have

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin(x)\sin(x)dx = \frac{1}{\pi}\int_0^\pi \frac{1-\cos(2x)}{2}dx = \frac{1}{\pi}\left(\frac{x - \sin(2x)}{2}\right)\Bigg|_{x=0}^{\pi} = \frac{1}{2}$$

Similarly, we have for $n \geq 2$

$$b_n = \frac{1}{\pi}\int_0^\pi \sin(x)\sin(nx)dx = \frac{1}{\pi}\int_0^\pi \frac{\cos((n-1)x) - \cos((n+1)x)}{2}dx$$
$$= \frac{1}{2\pi}\left(\frac{\sin((n-1))x}{n-1} - \frac{\sin((n+1)x)}{n+1}\right)\Bigg|_{x=0}^{\pi} = 0$$

Putting it all together the Fourier series is

$$\frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1} + \frac{\sin(x)}{2}$$

(c)

$$f(x) = \begin{cases} x^2 & 0 \leq x \leq \pi \\ 0 & -\pi \leq x < 0 \end{cases}$$

Solution: We calculate

$$a_0 = \frac{1}{\pi}\int_0^\pi x^2 dx = \frac{\pi^2}{3}$$

Next, we have (for $n \geq 1$)

$$
\begin{aligned}
a_n = \frac{1}{\pi}\int_0^\pi x^2\cos(nx)dx \\
&= \frac{1}{\pi}\int_0^\pi d\left(x^2\frac{\sin(nx)}{n} + 2x\frac{\cos(nx)}{n^2} - 2\frac{\sin(nx)}{n^3}\right) \\
&= \frac{1}{\pi}\left(x^2\frac{\sin(nx)}{n} + 2x\frac{\cos(nx)}{n^2} - 2\frac{\sin(nx)}{n^3}\right)\Bigg|_{x=0}^{\pi} \\
&= (-1)^n\frac{2}{n^2}
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
b_n = \frac{1}{\pi}\int_0^\pi x^2\sin(nx)dx \\
&= \frac{1}{\pi}\int_0^\pi d\left(-x^2\frac{\cos(nx)}{n} + 2x\frac{\sin(nx)}{n^2} + 2\frac{\cos(nx)}{n^3}\right) \\
&= \frac{1}{\pi}\left(-x^2\frac{\cos(nx)}{n} + 2x\frac{\sin(nx)}{n^2} + 2\frac{\cos(nx)}{n^3}\right)\Bigg|_{x=0}^{\pi} \\
&= -\pi\frac{(-1)^n}{n} + 2\frac{(-1)^n - 1}{n^3\pi}
\end{aligned}
$$

Putting it all together the Fourier series is

$$
\frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left((-1)^n\frac{2\cos(nx)}{n^2} + \left(-\pi\frac{(-1)^n}{n} + 2\frac{(-1)^n - 1}{n^3\pi}\right)\sin(nx)\right)
$$

2. Use the answers in the previous exercise to calculate the Fourier series for the following functions:

   (a)

$$
f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ -1 & -\pi \leq x < 0 \end{cases}
$$

Solution: If $f_0(x)$ denotes the function in Q1(a), then $f(x) = f_0(x) - f_0(-x)$. Since the constant term in the Fourier series will cancel out, and the $\sin(nx)-$

$\sin(-nx) = 2\sin(nx)$ this gives the Fourier series of $f(x)$ as

$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{(2k+1)}$$

(b)

$$f(x) = \begin{cases} \sin(x) & 0 \le x \le \pi \\ -\sin(x) & -\pi \le x < 0 \end{cases}$$

Solution: If $f_0(x)$ denotes the function in Q1(b), then $f(x) = f_0(x) + f_0(-x)$. Since $\sin(x) + \sin(-x) = 0$, and $\cos(nx) + \cos(-nx) = 2\cos(nx)$ this gives the Fourier series of $f(x)$ as

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1}$$

(c) $f(x) = x^2$ for $x \in [-\pi, \pi]$.

Solution: If $f_0(x)$ denotes the function in Q1(c), then $f(x) = f_0(x) + f_0(-x)$. So, as above we are left with twice the cosine series.

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4\cos(nx)}{n^2}$$

(d) $f(x) = x|x|$ for $x \in [-\pi, \pi]$.

Solution: If $f_0(x)$ denotes the function in Q1(c), then $f(x) = f_0(x) - f_0(-x)$. So, as above we are left with twice the sine series.

$$2\sum_{n=1}^{\infty}\left(-\pi\frac{(-1)^n}{n} + 2\frac{(-1)^n-1}{n^3\pi}\right)\sin(nx)$$

3. (Starred) Given $a < b$ in $[-\pi, \pi]$, Find the Fourier series for the functions given below:

(a)

$$f(x) = \begin{cases} 1 & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f(x) = \begin{cases} x & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

4. (Starred) Use the previous exercise with scaling and linearity to find the Fourier Series of any piecewise linear function.