Solutions to Assignment 6

1. Solve the following first-order equations with regular singularities. Also solve them by the method of separation of variables/exact differentials and compare the solutions.

$$x\frac{dy}{dx} = \frac{2}{5}y$$
$$x\frac{dy}{dx} = \left(\frac{2}{5} + x\right) y$$
$$x\frac{dy}{dx} = \left(\frac{2}{5} + \frac{1}{3}x\right) y$$

Solution: In each case we put $y = \sum_m a_m x^m$ as a formal sum with the understanding that m's need not be integers and that $a_m = 0$ for sufficiently negative m.

We obtain the equations (by equating coefficients of x^m) in each case:

$$m a_m = \frac{2}{5} a_m$$
$$m a_m = \frac{2}{5} a_m + a_{m-1}$$
$$m a_m = \frac{2}{5} a_m + \frac{1}{3} a_{m-1}$$

The first equation gives $a_m \neq 0$ if and only if $m = 2/5$. So the solution is $y = cx^{2/5}$ which is also pretty evident!

The second and third equations express a_m as a multiple of a_{m-1} *unless* $m = 2/5$. It follows that if $m - k \neq 2/5$ for *any* non-negative integer k, then, since $a_{m-k} = 0$ for k sufficiently large, so is a_m. In other words, the only non-zero a_m are for $m = k + 2/5$ for non-negative integers k. In those cases we obtain the identities

$$a_{k+2/5} = \frac{1}{k!} a_{2/5}$$
$$a_{k+2/5} = \frac{1}{3^k \cdot k!} a_{2/5}$$

This gives the solution $y = cx^{2/5} \exp(x)$ for the second equation and $y = cx^{2/5} \exp(x/3)$ for the third equation.

2. Solve the following second-order equations with regular singularities.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \frac{1}{9}y = 0$$
$$x^2\frac{d^2y}{dx^2} + x(1+x)\frac{dy}{dx} - \frac{1}{9}y = 0$$
$$x^2\frac{d^2y}{dx^2} + x(1+x)\frac{dy}{dx} - \left(\frac{1}{9}+x\right)y = 0$$

Solution: In each case we put $y = \sum_m a_m x^m$ as a formal sum with the understanding that m's need not be integers and that $a_m = 0$ for sufficiently negative m.

We obtain the equations (by equating coefficients of x^m) in each case:

$$m(m-1)a_m + ma_m - \frac{1}{9}a_m = 0$$
$$m(m-1)a_m + ma_m + (m-1)a_{m-1} - \frac{1}{9}a_m = 0$$
$$m(m-1)a_m + ma_m + (m-1)a_{m-1} - \frac{1}{9}a_m + a_{m-1} = 0$$

In each case, a_m is multiplied by $m^2 - (1/9)$ which zero if and only if $m = \pm 1/3$.

In the first equation, when $m^2 - (1/9)$ is non-zero, then $a_m = 0$. So the general solution takes the form $y = ax^{-1/3} + bx^{1/3}$, with $a_{-1/3} = a$ and $a_{1/3} = b$.

The second and third equations express a_m as a multiple of a_{m-1} *unless* $m^2 - 1/9 = 0$. It follows that if $m - k \neq \pm 1/3$ for *any* non-negative integer k, then, since $a_{m-k} = 0$ for k sufficiently large, so is a_m. In other words, the only non-zero a_m are for $m = k - 1/3$ and $m = k + 1/3$ for non-negative integers k.

So, in each case we put $b_k = a_{k-1/3}$ and $c_k = a_{k+1/3}$.

We then obtain the identities (for all $k \geq 0$) as follows (the first row is for the second equation and the second row is for the third equation):

$$b_{k+1} = -\frac{3k-4}{k(3k-2)}b_k \qquad c_{k+1} = -\frac{3k-2}{k(3k+2)}c_k$$
$$b_{k+1} = -\frac{3k-7}{k(3k-2)}b_k \qquad c_{k+1} = -\frac{3k-5}{k(3k+2)}c_k$$

This identities inductively define all b_k in terms of b_0 and all c_k in terms of c_0.

In each case the general solution is of the form

$$y(x) = x^{-1/3}\sum_{k=0}^{\infty} b_k x^k + x^{1/3}\sum_{k=0}^{\infty} c_k x^k$$

3. Find one Frobenius solution of the following second-order equations with regular singularities. If another Frobenius solution is possible, then find that as well. (A Frobenius solution is a solution in terms of powers of x for $x > 0$.)

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \frac{1}{4}y = 0$$
$$x^2\frac{d^2y}{dx^2} + x(1+x)\frac{dy}{dx} - \frac{1}{4}y = 0$$
$$x^2\frac{d^2y}{dx^2} + x(1+x)\frac{dy}{dx} - \left(\frac{1}{4} + x\right)y = 0$$

Solution: In each case we put $y = \sum_m a_m x^m$ as a formal sum with the understanding that m's need not be integers and that $a_m = 0$ for sufficiently negative m.

We obtain the equations (by equating coefficients of x^m) in each case:

$$m(m-1)a_m + ma_m - \frac{1}{4}a_m = 0$$
$$m(m-1)a_m + ma_m + (m-1)a_{m-1} - \frac{1}{4}a_m = 0$$
$$m(m-1)a_m + ma_m + (m-1)a_{m-1} - \frac{1}{4}a_m + a_{m-1} = 0$$

In each case, a_m is multiplied by $m^2 - (1/4)$ which zero if and only if $m = \pm 1/2$.

In the first equation, when $m^2 - (1/4)$ is non-zero, then $a_m = 0$. So the general solution takes the form $y = ax^{-1/2} + bx^{1/2}$, with $a_{-1/2} = a$ and $a_{1/2} = b$.

The second and third equations express a_m as a multiple of a_{m-1} *unless* $m^2 - 1/4 = 0$. It follows that if $m - k \neq \pm 1/4$ for *any* non-negative integer k, then, since $a_{m-k} = 0$ for k sufficiently large, so is a_m. In other words, the only possible non-zero a_m are for $m = k - 1/2$ and $m = k + 1/2$ for non-negative integers k.

Unlike the previous case, $(k - 1/2) = (k + 1/2) - 1$. So, we only obtain a solution in this form for $m = k + 1/2$ ($1/2$ is the *larger* of the two roots) with non-negative integers k. So, in each case we put $b_k = a_{k+1/2}$.

We then obtain the identities (for all $k \geq 0$) as follows (the first row is for the second equation and the second row is for the third equation):

$$b_{k+1} = -\frac{2k-1}{2k(k+1)}b_k$$
$$b_{k+1} = -\frac{2k-3}{2k(k+1)}b_k$$

This identities inductively define all b_k in terms of b_0.

In each case the general Frobenius solution is of the form

$$y(x) = x^{1/2} \sum_{k=0}^{\infty} b_k x^k$$

4. Given an equation $x^2 y'' + pxy' + qy = 0$, where p and q are (convergent) power series in x such that $(p(0) - 1)^2 > 4q(0)$. Find a substitution of the form $y = x^a z$ so that the equation for z has the form $x^2 z'' + rxz' + sz = 0$ where $r(0) = 1$ and $s(0) < 0$.

Solution: If we substitute y as required, we obtain

$$\begin{aligned} y' &= ax^{a-1}z + x^a z' \\ y'' &= a(a-1)x^{a-2}z + 2ax^{a-1}z' + x^a z'' \end{aligned}$$

Substituting this in the equation, we obtain

$$\left(a(a-1)x^a z + 2ax^{a+1}z' + x^{a+2}z''\right) + p\left(ax^a z + x^{a+1}z'\right) + qx^a y = 0$$

Dividing by x^a, we get

$$x^2 z'' + (2a + p)\, xz' + (a(a-1) + ap + q)\, z = 0$$

So we have $r = 2a + p$ and $s = a(a-1) + ap + q$. Which means that $r(0) = 2a + p(0)$ and $s(0) = a^2 + a(p(0) - 1) + q(0)$. If we put $a = (1 - p(0))/2$, then we check easily that $s(0) < 0$.

5. Given an equation $x^2 y'' + pxy' + qy = 0$, where p and q are (convergent) power series in x such that $p(0) \geq 1$ and $q(0) = 0$. Show that there is a power series in x which solves this equation.

Solution: The indicial equation for this equation is $m^2 + (p(0) - 1)m = 0$. It follows that $m_1 = 0$ is the larger root. Hence, the Frobenius solution is an ordinary power series.

One can also check this by directly substituting a power series $y = \sum_{k=0}^{\infty} a_k x^k$ in the above equation as a formal solution. We then check that we obtain an inductive formula for a_k in terms of a_{k-r} for $r \geq 1$ an integer. The denominator of this expression is $k^2 + (p(0) - 1)k$ which is (due to the hypothesis) larger than k^2. One can use this to prove the convergence.