Equations with regular singularities

1. Solve the following first-order equations with regular singularities. Also solve them by the method of separation of variables/exact differentials and compare the solutions.

$$x\frac{dy}{dx} = \frac{2}{5}y$$
$$x\frac{dy}{dx} = \left(\frac{2}{5} + x\right)y$$
$$x\frac{dy}{dx} = \left(\frac{2}{5} + \frac{1}{3}x\right)y$$

2. Solve the following second-order equations with regular singularities.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \frac{1}{9}y = 0$$
$$x^2\frac{d^2y}{dx^2} + x(1+x)\frac{dy}{dx} - \frac{1}{9}y = 0$$
$$x^2\frac{d^2y}{dx^2} + x(1+x)\frac{dy}{dx} - \left(\frac{1}{9} + x\right)y = 0$$

3. Find one Frobenius solution of the following second-order equations with regular singularities. If another Frobenius solution is possible, then find that as well. (A Frobenius solution is a solution in terms of powers of x for $x > 0$.)

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \frac{1}{4}y = 0$$
$$x^2\frac{d^2y}{dx^2} + x(1+x)\frac{dy}{dx} - \frac{1}{4}y = 0$$
$$x^2\frac{d^2y}{dx^2} + x(1+x)\frac{dy}{dx} - \left(\frac{1}{4} + x\right)y = 0$$

4. Given an equation $x^2y'' + pxy' + qy = 0$, where p and q are (convergent) power series in x such that $(p(0) - 1)^2 > 4q(0)$. Find a substitution of the form $y = x^a z$ so that the equation for z has the form $x^2z'' + rxz' + sz = 0$ where $r(0) = 1$ and $s(0) < 0$.

5. Given an equation $x^2y'' + pxy' + qy = 0$, where p and q are (convergent) power series in x such that $p(0) \geq 1$ and $q(0) = 0$. Show that there is a power series in x which solves this equation.