Differential Equations for Scientists

Kapil Hari Paranjape

Some examples of integral curves

Given a family $\Phi(x, y) = c$ of curves, one of the problems that quickly reduces to the problem of finding integral curves is that of finding the *orthogonal* family of curves. The given family of curves has the tangent $(\partial\Phi/\partial y, -\partial\Phi/\partial x)$ as the point (x, y). The orthogonal curves have tangents parallel to $(\partial\Phi/\partial x, \partial\Phi/\partial y)$. Equivalently, the differential of Φ is $(\partial\Phi/\partial x)dx + (\partial\Phi/\partial y)dy$. To find the orthogonal curves we need to solve the differential $(\partial\Phi/\partial y)dx - (\partial\Phi/\partial x)dy$.

In this set of solved examples we carry out such computations.

Circles

One of the simplest examples of solving for the integral curve associated with a differential is given by $xdx + ydy$. In this case we see that the flow takes the form

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x$$

The corresponding flow through (a, b) is given by

$$(x(t), y(t)) = (a\cos t + b\sin t, -a\sin t + b\cos t)$$

We see that this satisfies the equation $x^2 + y^2 = a^2 + b^2$. Hence, the flow curves are given by $\Phi(x, y) = c$ where $\Phi(x, y) = x^2 + y^2$. We could have also directly noticed that

$$\frac{\partial\Psi}{\partial x} = x$$
$$\frac{\partial\Psi}{\partial y} = y$$

has the solution $\Psi(x, y) = (x^2 + y^2)/2$.

By the theory worked out earlier, this tells us that, for *any* differential of the form \$ r x dx + r y dy\$ (with an *arbitrary* well-behaved function r), the function $\Phi(x, y) = x^2 + y^2$ solves the problem under consideration. In particular, we can solve the differentials$dx + (y/x)dy$ or $x^2ydx + xy^2dy$ by using the "integrating factors" x and $1/(xy)$ respectively.

Obviously, not all problems are so simple!

Ellipses

To illustrate the method, let us consider the problem of solving the differential $axdx + bydy$, in other words $(M, N) = (ax, by)$. We see that $(\partial N/\partial x) - (\partial M/\partial y) = 0$. So this is an "exact" differential. We then note that

$$\Phi(x, y) - \Phi(0, y) = \int_0^x M dx = ax^2/2$$

Next, we note that

$$\frac{d\Phi(0, y)}{dy} = by - \frac{\partial}{\partial y}(ax^2/2) = by$$

It follows that $\Phi(0, y) = by^2/2 + c$ for some constant c. This gives us $\Phi(x, y) = c + ax^2/2 + by^2/2$ as can also be found by "inspection".

Let us again note that the apparently more complicated looking differential \$ (a/y) dx + (b/x) dy\$ can also be solved with the same function using the integration factor $q = xy$ as we see by inspection.

Orthogonal curves

We now consider the problem of finding curves that are everywhere orthogonal to the ellipses $ax^2 + by^2 = c$. This is the problem of finding the curves whose tangents are *parallel* to $(2ax, 2by)$ at (x, y). In other words, we want to solve the differential $bydx - axdy$, so we put $(M, N) = (by, -ax)$.

We now check that $(\partial N/\partial x) - (\partial M/\partial y) = (a + b)$. So this is *not* exact when $a + b =\neq 0$.

When $a + b = 0$, we see easily that $\Phi(x, y) = bxy + c$ solve the problem.

So, let us consider the case $a + b \neq 0$. In this case, we note that

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{N} = \frac{a + b}{-ax}$$

is a function of x alone. So we put

$$q(x) = \exp\left(\int \frac{-b - a}{ax} dx\right) = x^{-b/a-1}$$

as the integrating factor. We note that

$$\frac{\partial}{\partial x}(x^{-b/a-1}N) = \frac{d}{dx}(-ax^{-b/a}) = bx^{-b/a-1}$$

Moreover,

$$\frac{\partial}{\partial y}(x^{-b/a-1}M) = \frac{\partial}{\partial y}(bx^{-b/a-1}y) = bx^{-b/a-1}$$

So the differential $qM dx + qN dy$ is closed. In fact, by inspection it is not difficult to check that $\Phi(x,y) = -ax^{-b/a}y$ satisfies

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= qM & &= bx^{-b/a-1}y \\ \frac{\partial \Phi}{\partial y} &= qN & &= -ax^{-b/a} \end{aligned}$$

Now the family of curves $\Phi(x,y) = c$ is the same as the family of curves $y^a = Cx^b$, and so it is more traditional to write the curves this way. We can also check that if $\Psi(x,y) = x^{-b}y^a$, then

$$\begin{aligned} \frac{\partial \Psi}{\partial x} &= rM \\ \frac{\partial \Phi}{\partial y} &= rN \end{aligned}$$

for a suitable function r when $(M,N) = (by, -ax)$.

As a final note, we did not seriously use the signs of a and b, so the same analysis also works for hyperbolas!