Existence of Solutions

(To simplify reading, these notes will use **boldface** symbols for vectors.)

One way to determine a solution is to determine its properties and use it to isolate the solution. So let us assume that $\mathbf{v}(t)$ is a solution of the initial value problem

$$\frac{d\mathbf{v}}{dt} = \mathbf{f}(\mathbf{v}) \text{ and } \mathbf{v}(t_0) = \mathbf{v}_0$$

By the fundamental theorem of calculus (applied entry-by-entry to $\mathbf{v}$), we get

$$\mathbf{v}(t) - \mathbf{v}(t_0) = \int_{t_0}^{t} \frac{d\mathbf{v}}{dt} dt = \int_{t_0}^{t} \mathbf{f}(\mathbf{v}) dt$$

This can be written as

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{v}) dt$$

which *looks* like a formula for $\mathbf{v}$ except that the right-hand side also depends on $\mathbf{v}$!

Instead, we can think of this as a map from vector-valued functions of t to vector-valued functions of t given by

$$\mathbf{w} \mapsto \Phi(\mathbf{w}) = \mathbf{u} \text{ where } \mathbf{u}(t) = \mathbf{w}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{w}) dt$$

In that case, we can think of the solution of the ODE as a *fixed* point of this map since the previous identity is recast as $\mathbf{v} = \Phi(\mathbf{v})$. (Recasting an equation as a fixed point problem is a very useful technique.)

We can start with a suitably chosen vector-valued function $\mathbf{v}^{(1)}$, and define $\mathbf{v}^{(k+1)} = \Phi(\mathbf{v}^{(k)})$. If we are lucky(!), this sequence $\mathbf{v}^{(k)}$ will *converge* to vector-valued function $\mathbf{v}$. Assuming that Φ is continuous in a suitable sense we will have the limiting identity

$$\mathbf{v} = \lim_{k\to\infty} \mathbf{v}^{(k+1)} = \lim_{k\to\infty} \Phi(\mathbf{v}^{(k)}) = \Phi\left(\lim_{k\to\infty} \mathbf{v}^{(k)}\right) = \Phi(\mathbf{v})$$

which is the equation we want.

For this approach to work, we need the condition that if $\mathbf{v} - \mathbf{w}$ is small then $\Phi(\mathbf{v}) - \Phi(\mathbf{w})$ should be small too. In our case, this second expression simplifies to

$$\int_{t_0}^{t} \left(\mathbf{f}\left(\mathbf{v}\right) - \mathbf{f}\left(\mathbf{w}\right)\right) dt$$

Now, we only want the solution in a fixed time interval $(t_0 - T, t_0 + T)$, so this integral is at most $2T$ times $\mathbf{f}\left(\mathbf{v}\right) - \mathbf{f}\left(\mathbf{w}\right)$. So the simplest condition we can ask for, is that there is a constant M so that

$$\|\mathbf{f}\left(\mathbf{v}\right) - \mathbf{f}\left(\mathbf{w}\right)\| \leq M\|\mathbf{v} - \mathbf{w}\|$$

where $\|\cdot\|$ denotes the length of the vector. This is called a Lipschitz condition on the function $\mathbf{f}$ in memory of the mathematician Rudolf Lipschitz.

In summary, suppose we want to solve the differential equation

$$\frac{d\mathbf{v}}{dt} = \mathbf{f}(\mathbf{v}) \text{ and } \mathbf{v}(t_0) = \mathbf{v}_0$$

Further assume that there is a $B > 0$ so that, for all $\mathbf{v}$ and $\mathbf{w}$ which are at a distance of at most B from $\mathbf{v}_0$, the Lipschitz condition

$$\|\mathbf{f}\,(\mathbf{v}) - \mathbf{f}\,(\mathbf{w})\| \leq M\|\mathbf{v} - \mathbf{w}\|$$

is satisfied for a suitable positive constant M. We now pick T so that $2T/M < \min\{B, 1\}/2$ and define:

1. $\mathbf{v}^{(1)}(t) = \mathbf{v}_0$ for all t.
2. $\mathbf{v}^{(k+1)} = \Phi(\mathbf{v}^{(k)})$.

Then, one can show, by the usual methods of analysis that $\mathbf{v}^{(k)}$ converges, for all t in $(t_0 - T, t_0 + T)$, to a vector-valued differentiable function $\mathbf{v}$ which satisfies the differential equation. In fact, we can replace $\mathbf{v}^{(1)}$ by *any* continuous vector-valued function in $(t_0 - T, t_0 + T)$ which has value $\mathbf{v}_0$ at t_0 and has values within a distance of B from this vector. This second assertion shows that the solution is unique.

We can also vary the initial vector $\mathbf{v}_0$ and show that the solution varies continuously as we do so. This gives us the Picard-Lindelöf theorem.