Solutions to Quiz 2

(5 marks) 1. Give the general solution of the differential equation

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = y$$

by the method using eigenvectors of a suitable matrix.

Solution: We see that the equation is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

where the matrix A is $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. This has characteristic polynomial $(T-2)(T-1) = 0$ which has the solutions $T = 2$ and $T = 1$. (1 Mark for this step.)

We find the eigenvectors of this by writing

$$2\mathbf{1}_2 - A = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$$

By inspection we see that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is sent to 0 by this matrix. Similarly, by writing

$$\mathbf{1}_2 - A = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

we see that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is sent to 0 by this matrix. Thus, if we put

$$G = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Then we have $A \cdot G = G \cdot D$ where D is the diagonal matrix with entries 2 and 1. (1 Mark for each eigenvector or 2 marks for matrix G.)

Putting $\begin{pmatrix} u \\ v \end{pmatrix} = G^{-1} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ we see that

$$\frac{du}{dt} = 2u$$
$$\frac{dv}{dt} = v$$

These have the solutions $u = ae^{2t}$ and $v = be^t$. (1 Mark for this step.) It follows that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} ae^{2t} + be^t \\ -be^t \end{pmatrix}$$

(1 Mark for this final solution.)

(4 marks for obtain the correct solution *without* reducing to the separate diagonal form.)