Solutions to Assignment 1

1. A body is falling and experiencing air-drag which is proportional to the velocity. In other words, that the acceleration is $a(t) = mg - rv(t)$, where $v(t)$ is the velocity at time t and $a(t)$ is the acceleration at time t. Formulate this law as an ordinary differential equation for the height h.

Solution: We have $a(t) = dv/dt$. Hence, the equation is $dv/dt = mg - rv$. Since v is the downward velocity, we have $v = -dh/dt$ and $a = -d^2h/dt^2$. Thus the differential equation for the height is

$$-\frac{d^2h}{dt^2} = mg - r\frac{dh}{dt}$$

2. A naive formulation of population dynamics is as follows. The population is divided into three groups u, v and w consisting of under-age people, old people and people of a reproductive age. The rate at which the population u grows is proportional to w. The rate at which the population w grows is proportional to u (since some proportion of under-age people becomes "old enough") minus some multiple of w since some proportion of people grow too old. The rate at which the population v changes is some linear combination of w (since some people become too old) and v (since some old people die). Formulate this as an ordinary differential equation for (u, v, w) as a function of time t.

Solution: We have $du/dt = a \cdot w$ for some positive constant a (which represents the rate at which births take place). The next equation is $dw/dt = bu - cw$, where b is the fraction of the under-age population that attain reproductive age and c is the proportion of the population of reproductive age who become too old. Finally $dv/dt = cw - dv$ with cw as above and d being the rate at which old people die. In summary, this can be formulated as

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} aw \\ cw - dv \\ bu - cw \end{pmatrix}$$

3. A constant magnetic field is described by a vector $\vec{B}$. The acceleration experienced by a charged particle moving with velocity $\vec{v}$ is given by $\vec{B} \times \vec{v}$ where $\times$ denotes the "cross"-product between two vectors. Formulate this as an ordinary differential equation for the velocity $\vec{v}$. Also formulate this as an ordinary differential equation for the 6 dimensional vector $(\vec{x}, \vec{v})$, where $\vec{x}$ denotes the position vector.

Solution: The first condition is restated as $d\vec{v}/dt = \vec{B} \times \vec{v}$. It follows that the comlete differential equation is

$$\frac{d}{dt}\begin{pmatrix}\vec{x}\\ \vec{v}\end{pmatrix} = \begin{pmatrix}\vec{v}\\ \vec{B} \times \vec{v}\end{pmatrix}$$

4. A mathematician is interested in the problem of finding all differentiable functions $f : \mathbb{R} \to \mathbb{C} - \{0\}$ such that $f(0) = 1$ and $f(x + y) = f(x)f(y)$. Formulate this as a problem in ordinary differential equations. (Note that the condition is that of finding all differentiable group homomorphisms from the additive group of real numbers to the multiplicative group of complex numbers.)

Solution: By differentiability, we see that

$$\frac{df}{dz}_{z=x} = \lim_{t\to 0} \frac{f(x+t) - f(x)}{t}$$

We can now calculate this as

$$\frac{f(x+t) - f(x)}{t} = \frac{f(x)f(t) - f(x)}{t} = \frac{f(t) - f(0)}{t} \cdot f(x) = \frac{df}{dz}_{z=0} \cdot f(x)$$

Thus, the differential equation satisfied is $(df/dx) = cf(x)$ where c is a constant.

5. Given that the gravitational constant is 1 and the mass of the earth, sun and moon are a, b and c, formulate the dynamical problem of the motion of these bodies (treated as point particles) as an ordinary differential equation using Newton's law of gravitation and Newton's first law of motion. (Note that this is an equation in 18 dimensional space as there are 3 bodies each of which have a position and a velocity!)

Solution: Let $\vec{x}$, $\vec{y}$, $\vec{z}$ denote the positions of the earth, sun and moon and let $\vec{u}$, $\vec{v}$, $\vec{w}$ denote their velocities. By the law of gravitation and Newton's first law of motion, we have

$$\frac{d\vec{u}}{dt} = b\frac{\vec{y} - \vec{x}}{\|\vec{y} - \vec{x}\|^3} + c\frac{\vec{z} - \vec{x}}{\|\vec{z} - \vec{x}\|^3}$$

Similarly, for the sun and the moon. We also have $d\vec{x}/dt = \vec{u}$. Putting all these equations together, we get the required equations.

6. Given a hill whose shape is given by $z = f(x, y)$. A ball rolls along the hill (under gravity) starting at a point $\vec{p}$ on the hill with an initial velocity $\vec{v}$ (which is tangent to the surface). Assuming that there is no rolling friction, formulate the problem of the motion of the ball as a differential equation. (Assume that the normal force *constrains* the ball to stay on the surface.)

More?!: Please see the *Miscellaneous problems at the end of Chapter 1* in the book on "Differential Equations" by G. F. Simmons.