(2 marks) 1. If $a_n \geq 0$ is a sequence such that $\sum_n a_n$ diverges, then does $\sum_n a_n \cdot 1/(n^2 + n^2 a_n)$ also diverge?

Solution: Since $a_n \geq 0$ we have $0 \leq a_n/(1 + a_n) \leq 1$. It follows that

$$0 \leq a_n \cdot 1/(n^2 + n^2 a_n) \leq 1/n^2$$

So the sum converges.

(2 marks) 2. If $a_n \geq 0$ is a sequence such that $\sum_n a_n$ diverges, then is there a sequence (b_n) converging to 0 such that $\sum_n a_n b_n$ also diverges?

Solution: Since $\sum_n a_n$ diverges, there is a subsequence n_k of integers so that (putting $n_0 = 0$), we have for each positive integer k,

$$\sum_{n=n_{k-1}+1}^{n=n_k} a_n \geq k$$

We now define $b_n = 1/k$ for $n_{k-1} + 1 \leq n \leq n_k$. Then

$$\sum_{n=1}^{n_k} a_n b_n = \sum_{r=1}^{k} \left(\sum_{n=n_{r-1}+1}^{n=n_r} a_n \right) \cdot 1/r \geq \sum_{r=1}^{k} r \cdot 1/r = k$$

Which goes to infinity with k. On the other hand (b_n) converges to 0.

(2 marks) 3. Is there a norm $\|\cdot\|$ on $\mathbb{R}^\infty$ for which it is a complete metric space?

Solution: By the Baire category theorem, a complete metric space cannot be the union of proper closed subspaces. On the other hand $\mathbb{R}^\infty = \bigcup \mathbb{R}^n$. Since a finite dimensional space is complete with respect to *any* norm, $\mathbb{R}^n$ for each n is a proper closed subspace. Thus the union cannot be complete.

(2 marks) 4. Is there a normed linear space V for which the (topological) closure of the open unit ball is *smaller* than the closed unit ball?

Solution: If v is a vector with $\|v\| \leq 1$ then define $v_n = (1 - 1/n)v$, we see that $\|v_n\| < 1$ and it converges to v. Conversely, if w_n is any sequence of vectors such that $\|w_n\| < 1$ and it converges to w then, by continuity of norm $\|w\| \leq 1$. So the topological closure of the open unit ball is the closed unit ball.

(2 marks) 5. Let V denote the space of all bounded sequences (a_n) such that (a_{2n}) is a convergent sequence. Put the "sup" norm on V. Is V a Banach space?

> **Solution:** We note that V there is a map $V \to \ell_\infty \oplus \mathcal{C}$ given by
>
> $$(a_n) \mapsto ((a_{2n-1}), (a_{2n})) = ((b_n), (c_n))$$
>
> This map is 1-1 and onto. Moreover, the norm on V is the same as the norm on the right hand side given by $\max\{\|(b_n)\|, \|(c_n)\|\}$. This latter norm is complete. So V is a Banach space as well.

(2 marks) 6. Given unit norm vectors v_n in a normed linear space V, define $M : \mathbb{R}^\infty \to V$ by $(a_n) \mapsto \sum_n a_n v_n$. (Note that the sum is finite!) Is M continuous with the ℓ_1 norm on $\mathbb{R}^\infty$?

> **Solution:** We have
>
> $$\|M((a_n))\| = \|\sum_n a_n v_n\| \leq \sum_n |a_n| \|v_n\| \leq \|(a_n)\|_1$$
>
> Hence, we have $\|M\| \leq 1$.

(2 marks) 7. Is there a linear functional $f : \mathbb{R}^\infty \to \mathbb{R}$ continuous with respect to ℓ_2-norm such that $\|f\| = 1$ and $|f(e_k)| < 1$ for all k? (Here e_k denotes the "standard" basis of $\mathbb{R}^\infty$.)

> **Solution:** Take any sequence (a_n) of positive real numbers such that $\sum_n a_n^2 = 1$. (For example $a_n = 2^{-n/2}$.) Then (a_n) is in ℓ_2. Hence, the linear functional f given by
>
> $$f(v) = \langle v, (a_n) \rangle$$
>
> is continuous and $\|f\| = 1$. On the other hand $f(e_k) = a_k < 1$.

(2 marks) 8. Let $\mathcal{C}$ be the space of convergent sequences with "sup" norm and $\mathcal{C}_0$ be the subspace of sequences converging to 0. Is there a continuous linear functional $f : \mathcal{C} \to \mathbb{R}$ such that f is 0 on $\mathcal{C}_0$ and $f((1)) = 1$, where (1) denotes the sequence all of whose entries are 1?

> **Solution:** We define $f((b_n)) = \lim_n b_n$. This is a continuous linear functional on $\mathcal{C}$ with the required properties.

(2 marks) 9. Is there a 1-1 onto linear operator $T : \mathcal{C} \to \mathcal{C}_0$ which is continuous?

Solution: We define $S : \mathcal{C}_0 \to \mathcal{C}$ by

$$S((a_n)) = (a_1 + a_2, a_1 + a_3, \dots)$$

Then S is clearly 1-1 and continuous. Given any convergent sequence (b_n) let b be its limit, then it is clear that the sequence

$$(a_n) = (b, b_1 - b, b_2 - b, \dots) \text{ converges to } 0$$

Moreover, it is clear that $S((a_n)) = (b_n)$. So S is onto. Since these are Banach spaces, it has a continuous inverse T.

(2 marks) 10. Is there a 1-1 onto linear operator $T : (\mathbb{R}^\infty, \ell_2) \to (\mathbb{R}^\infty, \ell_1)$ which is not continuous? (The notation means that we take the ℓ_2 norm on the left and the ℓ_1 norm on the right.)

Solution: The identity operator is 1-1 and onto. However, we see that if $v_k = (a_{n,k})$ where

$$a_{n,k} = \begin{cases} 1/n & \text{for } n \leq k \\ 0 & \text{for } n > k \end{cases}$$

Then v_k lies in $\mathbb{R}^\infty$ and has bounded ℓ_2 norm but not bounded ℓ_1 norm.

(2 marks) 11. Is there a 1-1 onto linear operator $T : \ell_1 \to \ell_2$ which is continuous?

Solution: There is no such operator since if there were such an operator then its inverse would also be continuous since both are Banach spaces. On the other hand ℓ_1 and ℓ_2 cannot be isomorphic since the dual of ℓ_1 is ℓ_∞ which is not *separable*, while ℓ_2 is separable and its own dual.

(2 marks) 12. Is there a 1-1 linear operator $T : \ell_2 \to \ell_2$ which is also compact?

Solution: The operator $T((a_n)) = (a_n/n)$ is 1-1. If we define T_k to be the truncated operator which maps to sequences which are 0 beyond k, then

$$(T - T_k)((a_n)) = (0, \dots, 0, a_{k+1}/(k+1), \dots)$$

We have

$$\|(T-T_k)((a_n))\|_2 = \left(\sum_{r=k+1}^{\infty} |a_r|^2/r^2\right)^{1/2} \leq \|((a_n))\|_2/(k+1)$$

which 0 as k goes to infinity. Thus, T_k converges to T in operator norm. Since T_k are finite rank operators, the operator T is compact.

(2 marks) 13. Let ℓ_∞ be the space of all bounded sequences with "sup" norm. Define $f : \ell_\infty \to \mathbb{R}$ by $f((x_n)) = \limsup x_n$. Is this is a continuous linear functional?

Solution: This is not even linear!

$$f(-(x_n)) = f((-x_n)) = -\liminf x_n$$

So $f(((-1)^n)) = 1$ but $f(-((-1)^n)) = 1 \neq -1$!

(2 marks) 14. Is there a non-zero continuous function f on $[0,1]$ such that $\int_0^1 x^n f(x)dx = 0$ for all n?

Solution: (During the exam it was clarified that $n \geq 0$ and it was permitted to assume that f is real valued.)

From the given condition it follows that $\int_0^1 P(x)f(x)dx = 0$ for *all* polynomials $P(x)$. Since polynomials are dense in $C[0,1]$ with the "sup" norm, we can for any $\epsilon > 0$ find a $P(x)$ polynomial such that $|P(x) - f(x)| < \epsilon$ for all x in $[0,1]$. It follows that

$$\left|\int_0^1 f(x)^2 dx\right| = \left|\int_0^1 \left(P(x)f(x) - f(x)^2\right) dx\right|$$
$$\leq \int_0^1 |P(x)f(x) - f(x)^2| dx \leq \epsilon \int_0^1 |f(x)| dx$$

Since $\int_0^1 |f(x)|dx$ is a fixed constant independent of ϵ, we see that $\int_0^1 f(x)^2 dx = 0$.

Since this is the integral of a non-negative continuous function it cannot be 0 unless the function is 0. (Here we have assumed that the function is real valued. A similar argument can be given for a complex valued function by approximating $\overline{f}$ by polynomials.)

(2 marks) 15. Consider the operator $T : \ell_2 \to \ell_2$ given by $T((a_n)) = (a_n/(1+n)^2)$. Is there a non-zero linear functional such that $f(v) = f(T(v))$ for all v in ℓ_2?

Solution: (It was implicit that f is *continuous*!)

We note that T is a compact operator (for example, since it is diagonal with eigenvalues $1/(1+n)^2$ with go to 0). Hence 1 is not an eigenvalue. It follows that $T - 1$ is 1-1 and thus onto as well. It follows that any f as above *must* be 0 on all of ℓ_2.

Alternatively, we note that if e_k denotes the standard basis of ℓ_2), then $T(e_k) = e_k/(1+k)^2$. So, if $f(e_k) = f(T(e_k))$, then $f(e_k) = f(e_k)/(1+k)^2$ for all k. This means that $f(e_k) = 0$ for all k. We can use Riesz representation theorem to say that a linear functional is determined by its values on e_k for all k and so $f = 0$.

(2 marks) 16. Is the operator $T : \ell_1 \to \ell_1$ given by $(a_n) \mapsto ((1 - 1/n)a_n)$ a bounded operator?

Solution: Since $(1 - 1/n)$ are bounded, we have seen that this diagonal operator is bounded.

(2 marks) 17. Is the operator $T : \ell_1 \to \ell_1$ given by $(a_n) \mapsto (a_n/(n+1))$ an open operator?

Solution: It is clear that the given operator is 1-1. So if it were to be open, then its inverse would also be continuous. On the other hand it is clear that the inverse operator is not bounded.

(2 marks) 18. Is the operator $T : \ell_1 \to \ell_1$ given by $(a_n) \to (a_{n+1}/n^2)$ a compact operator?

Solution: This is an example of a nuclear operator and thus it is a compact operator. More specifically, let T_k be the truncated operator

$$T_k((a_n)) = (a_2, a_3/4, a_4/9, \ldots, a_{k+1}/k^2, 0, 0, \ldots)$$

We note that if $S_k = T - T_k$) then

$$S_k((a_n)) = (0, \ldots, 0, a_{k+2}/(k+1)^2, \ldots)$$

It follows that

$$\|S_k((a_n))\| \leq \|(a_n)\|_1/(k+1)^2$$

which goes to 0 as k goes to infinity. Hence, T is the limit of finite rank operators T_k. Thus it is compact.

(2 marks) 19. Is the image of operator $T : \ell_1 \to \ell_1$ given by $(a_n) \to (a_n - a_{n+1}/n^2)$ a closed subspace?

Solution: The operator has the form $1 - S$ where S is a compact operator (as seen in the previous problem).

Hence it is a Fredholm operator and its image is closed.

(2 marks) 20. Given a sequence of unit vectors e_n in a Banach space B, is there a continuous linear functional $f : B \to \mathbb{R}$ for which $f(e_n) \neq 0$ for *all* n?

Solution: (During the exam it was clarified that B is *infinite* dimensional but it is not required!)

For each n, by the Hahn-Banach theorem, there is a continuous linear functional f_n such that $f_n(e_n) = 1$. Hence, the subspace V_n of B^* that consists of continuous linear functionals g such that $g(e_n) = 0$ is a *proper* closed subspace of B^*. By the Baire category theorem, the union of V_n does not exhaust B. Hence, there is an element f in B which is not in *any* V_n. So $f(e_n) \neq 0$ for all n.