Solutions to Quiz 8

1. Consider the operator $T : \ell_2 \to \ell_2$ given by

$$T(a_1, a_2, \dots) = (c_1 a_1, c_2 a_2, \dots)$$

Answer the following questions:

(1 mark) (a) When is T bounded?

(1 mark) (b) When is T compact?

(1 mark) (c) What is the adjoint of T?

(1 mark) (d) When is T normal?

(1 mark) (e) When is T Fredholm?

Solution: We note that the eigenvalues of T are c_n. So for T to be bounded it is *necessary* that (c_n) lies in ℓ_∞ (the collection of bounded sequences). We also note that

$$\sum_n |c_n a_n|^2 \leq \|(c_n)\|_\infty^2 \sum_n |a_n|^2$$

In other words, $\|T\| \leq \|(c_n)\|_\infty$. Hence, it is also sufficient that (c_n) is in ℓ_∞. In other words:

> T is bounded if and only if (c_n) lies in ℓ_∞ and in this case $\|T\| \leq \|(c_n)\|_\infty$.

We note in passing that, if $c_n \neq 0$ for all n, then the inverse operator is given by

$$T^{-1}(a_1, a_2, \dots) = (a_1/c_1, a_2/c_2, \dots)$$

For T to be invertible T^{-1} needs to be bounded. Hence,

> T is invertible if and only if there is a positive constant $k > 0$ so that $|c_n| > k$ for all n.

Now let us consider the question of when T is compact. We know that for any $\epsilon > 0$ there are only *finitely* many eigenvalues of a compact operator that lie *outside* the disk of radius ϵ around 0. So, a necessary condition for T to be compact is that for all $\epsilon > 0$ there are only finitely many n for which $|c_n| > \epsilon$. In other words, there is an N so that $|c_n| \leq \epsilon$ for all $n > N$. Equivalently c_n converges to 0 as n goes to infinity. Thus, a *necessary* condition for T to be compact is that (c_n) lies in $\mathcal{C}_0$ (the collection of sequences converging to 0). If (c_n) lies in $\mathcal{C}_0$, let us define T_N by "truncation" of T as

$$T_N(a_1, a_2, \dots) = (c_1 a_1, c_2 a_2, \dots, c_N a_N, 0, 0, \dots)$$

It is clear that T_N is a finite rank operator. Moreover,

$$(T - T_N)(a_1, a_2, \dots) = (0, 0, \dots, 0, c_{N+1}a_{N+1}, c_{N+2}a_{N+2}, \dots)$$

Hence T_N converges to T in operator norm (by the norm calculation above). Thus for T to be compact it is also *sufficient* that (c_n) lies in $\mathcal{C}_0$.

> T is compact if and only if (c_n) lies in $\mathcal{C}_0$.

If $\{e_n\}$ is the standard Hilbert basis of ℓ_2, then we have $\langle Te_n, e_m \rangle = c_n \delta_{n,m}$. It follows easily that the adjoint of T is given by the sequence $(\overline{c_n})$, i. e.

$$T^*(a_1, a_2, \dots) = (\overline{c_1}a_1, \overline{c_2}a_2, \dots)$$

This identity makes it clear that T and T^* commute. So T is *always* normal.

> T is always normal.

Finally, for T to be Fredholm, its kernel has to be finite dimensional and its image has to be closed and have finite codimension. The kernel of T is spanned by those e_n for which $c_n = 0$. Hence, for T to be Fredholm, it is necessary that there are at most *finitely* many n for which $c_n = 0$. However, this is *not* sufficient. Let H be the subspace of ℓ_2 which is the orthogonal complement of the kernel of T; the collection $\{e_n : c_n \neq 0\}$ is a Hilbert basis of H. It is clear that kernel of T^* is the same as the kernel of T. Hence, H is also the orthogonal supplement to the kernel of T^*. In order for the image of T to be closed it is sufficient that it is equal to H. In other words, we want a condition so that T is *invertible* when restricted to H. As seen above this means that there is a positive constant $k > 0$ so that $|c_n| > k$ for all n such that $c_n \neq 0$. Combining these two conditions, we see that

> T is Fredholm if and only if there is a positive constant $k > 0$ and an integer N so that $|c_n| > k$ for all $n > N$.