Spectral Theorem for Normal Operators

Given an operator $S : V \to V$ we have defined the transpose $S^t : V^* \to V^*$.

If $V = H$ is a Hilbert spaces, then we have a natural map $\Lambda : H \to H^*$ given by $\Lambda(v)(w) = \langle w, v\rangle$. We note that Λ is $\mathbb{C}$ conjugate-linear in the sense that

$$\Lambda(z \cdot v) = \overline{z} \cdot \Lambda(v)$$

The Riesz representation theorem is the statement that Λ is onto. (It is easily seen to be one-to-one.)

Given a complex vector space V, we define $\overline{V}$ to be the same set with a new scalar multiplication $z \odot v = \overline{z} \cdot v$.

Exercise: Check that $\overline{V}$ becomes a complex vector space with the above scalar multiplication. (Addition is the same as in V.)

Exercise: Check that if V is a normed linear space, then so is $\overline{V}$. (Using the same norm as before.)

It follows that $\overline{H^*}$ is a vector space and we can think of Λ as an isomorphism $\tau : H \to \overline{H^*}$.

Exercise: Given a $\mathbb{C}$-linear operator $S : V \to W$, check that $S(z \odot v) = z \odot S(v)$.

It follows that we get a $\mathbb{C}$-linear operator $\overline{S} : \overline{V} \to \overline{W}$ given by the *same* underlying set map. The "adjoint" of an operator $S : H \to H$ is

$$S^* = \tau^{-1} \circ S^t \circ \tau : H \to H$$

Exercise: Check the identity $\langle Sv, w\rangle = \langle v, S^* w\rangle$ for all v, w in H.

In many texts, this identity is used to *define* the operator S^*.

An operator S is called *Hermitian* if $S = S^*$. This definition is often extended to "unbounded" operators (which we are not studying in detail in this course) and in that case, the term *self-adjoint* is also used.

Exercise: Check that an operator S is Hermitian if and only if $\langle Sv, w\rangle = \langle v, Sw\rangle$ for all v, w in H.

Exercise: Given *any* operator $T : H \to H$ define $\Re(T) = (T + T^*)/2$ and $\Im(T) = (T - T^*)/2\sqrt{-1}$. Show that $\Re(T)$ and $\Im(T)$ are Hermitian operators.

Note that $T = \Re(T) + \sqrt{-1}\Im(T)$ and $T^* = \Re(T) + \sqrt{-1}\Im(T)$.

Exercise: Given *any* operator $T : H \to H$, show that $T \circ T^*$ and $T^* \circ T$ are Hermitian operators.

An operator U is called *unitary* is $U \circ U^* = U^* \circ U = \mathbf{1}$.

Exercise: Check that an operator U is unitary if and only if $\langle Uv, Uw\rangle = \langle v, w\rangle$ for all v, w in H and U is onto.

Note that the right shift operator $R : \ell_2 \to \ell_2$ satisfies $\langle Rv, Rw\rangle = \langle v, w\rangle$ but R is not onto and so it is not unitary!

An operator is called *normal* if $S \circ S^* = S^* \circ S$; in other words, if S and S^* commute.

Exercise: If S is Hermitian and U is unitary and $S \circ U = U \circ S$, then check that $\circ U$ is a normal operator.

Exercise: With notation as in a previous exercise, show that S is a normal operator if and only if $\Re(S)$ and $\Im(S)$ commute.

We first will analyse the eigenvalues and eigenvectors of a single Hermitian operator and then extend these ideas to commuting collections of Hermitian operators.

Eigenvalues and Eigenvectors of Hermitian operators

Recall that $v \in H$ is called an eigenvector for $S : H \to H$ with eigenvalue λ if $S(v) = \lambda v$. Usually it is also assumed that $v \neq 0$. However, we shall only say that λ *is* an eigenvalue of S if there is a *non-zero* vector V which is an eigenvector of S with eigenvalue λ.

Exercise: For a Hermitian operator $S : H \to H$ and v a vector in H, show that $\langle Sv, v\rangle$ is a real number.

Exercise: If S is a Hermitian operator and λ is an eigenvalue, then show that λ is a real number. (Hint: For a non-zero eigenvector v with eigenvalue λ note that $\lambda\langle v, v\rangle = \langle Sv, v\rangle$.)

Exercise: If $S : H \to H$ is a Hermitian operator and v is an eigenvector, show that $S(v^\perp) \subset v^\perp$, where $v^\perp$ is the subspace of H consisting of vectors orthogonal to v.

Given a non-zero vector v, we define $\pi_v : H \to H$ as

$$\pi_v(w) = \frac{\langle w, v\rangle}{\langle v, v\rangle} \cdot v$$

Exercise: If v is a non-zero eigenvector of a Hermitian operator S, then show that $\pi_v \circ S = S \circ \pi_v$.

Exercise: If w is a non-zero eigenvector the Hermitian operator S for a *different* eigenvalue μ then show that $\pi_v(w) = 0$; equivalently, show that $\langle w, v\rangle = 0$.

The norm as an eigenvalue

In the finite-dimensional case, one proves (by a compactness argument) that the maximum of $\langle v, Sv\rangle$ for a Hermitian operator S is an eigenvalue. We will now give a similar argument can be used to show that the norm of a *compact* Hermitian operator is the absolute value of one of its eigenvalues.

As a consequence of the Riesz representation theorem we showed that for a vector v in a Hilbert space H,

$$\|v\| = \|\lambda(v)\| = \sup\{|\langle w, v\rangle| : \|w\| = 1\}$$

Exercise: For an operator $S : H \to H$, show that

$$\|S\| = \sup\{|\langle w, Sv\rangle| : \|v\| = 1 = \|w\|\}$$

Exercise: Show that

$$\sup\{|\langle w, Sv\rangle| : \|v\| = 1 = \|w\|\} = \sup\{\Re(\langle w, Sv\rangle) : \|v\| = 1 = \|w\|\}$$

where $\Re(z)$ denotes the real part of a complex number z. (Hint: Note that the following set is closed under multiplication by complex numbers of absolute value 1.)

$$\{\langle w, Sv\rangle : \|v\| = 1 = \|w\|\}$$

Now, if $S : H \to H$ is a Hermitian operator, then

$$\Re(\langle w, Sv\rangle) = \frac{1}{2}\left(\langle w, Sv\rangle + \langle Sv, w\rangle\right) = \frac{1}{2}\left(\langle w, Sv\rangle + \langle v, Sw\rangle\right)$$

On the other hand we have

$$\langle (v \pm w), S(v \pm w)\rangle = (\langle v, Sw\rangle + \langle w, Sw\rangle) \pm (\langle w, Sv\rangle + \langle v, Sw\rangle)$$

This gives us the identity

$$\Re(\langle w, Sv\rangle) = \frac{1}{4}\left(\langle v + w, S(v + w)\rangle - \langle v - w, S(v - w)\rangle\right)$$

Let use put $c = \sup\{|\langle v, Sv\rangle| : \|v\| = 1\}$. We then get

$$\Re(\langle w, Sv\rangle) \le \frac{c}{4}\left(\|v + w\|^2 + \|v - w\|^2\right)$$

Exercise: Show that $\|S\| \le c$. (Hint: Use the above calculations together with the parallelogram law as below.)

$$\|v + w\|^2 + \|v - w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right)$$

Exercise: Show that for a Hermitian operator $S : H \to H$ on a Hilbert space

$$\|S\| = \sup\{|\langle v, Sv\rangle| : \|v\| = 1\}$$

Exercise: Show that there is a sequence of unit vectors v_n in H so that $\langle v_n, Sv_n\rangle$ converges to d where $d^2 = \|S\|^2$. (Hint: We have sequence of real numbers whose absolute value converges, then a subsequence of these converges either on the positive side or the negative side!)

We can now calculate

$$\|Sv_n - dv_n\|^2 = \|Sv_n\|^2 + d^2\|v_n\|^2 - 2d\|Sv_n, v_n\| \leq d^2 + d^2 - 2d\|Sv_n, v_n\|$$

The last term converges to $d^2 + d^2 - 2d^2 = 0$ as n goes to infinity! So $(S - d\mathbf{1})v_n$ converges to 0 as n goes to infinity.

Now assume that $S : H \to H$ is a *compact* Hermitian operator. Then Sv_n contains a convergent subsequence, so we replace (v_n) by this subsequence. Let v be the limit of the sequence (Sv_n). Then $dv_n = Sv_n - (S - d\mathbf{1})(v_n)$ also converges to v since the second term converges to 0. It follows that Sv which is the limit of $S(dv_n) = dSv_n$ which is dv. In other words, we have shown that $Sv = dv$ so that v is an eigenvector of S with eigenvalue d.

Exercise: If $\|S\| \neq 0$ then show that v is non-zero. (Hint: Note that dv_n converges to v and v_n are unit vectors.)

Hence, if S is a compact Hermitian operator on a Hilbert space, then either $S = 0$ or, there is an eigenvalue d of S such that $d^2 = \|S\|^2$.

Structure of a Compact Hermitian Operator

In this section we will show that there is a basis of unit eigenvectors for a compact Hermitian operator $S : H \to H$.

Suppose that λ is a non-zero eigenvalue of S. Since $S - \lambda\mathbf{1}$ is a Fredholm operator, the subspace $V_\lambda = \ker(S - \lambda\mathbf{1})$ is finite dimensional. From the results above, we see that S maps $V_\lambda^\perp$ to itself. It is clear that the restriction of S to this closed subspace is also compact and Hermitian. Moreover, λ cannot be an eigenvalue for this restriction since *all* all eigenvectors of S with eigenvalue λ are in V_λ.

We have show that there are only finitely many eigenvalues outside $\{z : |z| < 1/n\}$ Hence, the non-zero eigenvalues of S form a countable (or possibly even finite!) sequence (λ_n) of real numbers. The subspace $V_k = V_{\lambda_n}$ of eigenvectors for eigenvalue λ_n is finite.

Let V_0 be the subspace of H that is the intersection of all the spaces $V_k^\perp$. This is a closed subspace and the restriction of S to V_0 has *no* non-zero eigenvalues.

Exercise: Show that the restriction of S to H_0 is 0. (Hint: We saw above that a non-zero compact Hermitian operator has a non-zero eigenvalue.)

Now let $(e_{n,k})_{k=1}^{r_n}$ be an orthonormal basis of V_n for $n \geq 1$ and let $e_{0,\alpha})_{\alpha \in A}$ be an orthonormal Hilbert basis of V_0. (Here we have allowed for A to be an infinite and perhaps even uncountable set!). Together, this collection gives a Hilbert basis of H which consists of eigenvectors of S as required.

Simultaneous Diagonalisation

We now want to extend the above result to compact *normal* operators.

As seen above a normal operator S can be written as $\Re(S) + \sqrt{-1}\Im(S)$ where $\Re(S)$ and $\Im(S)$ commute with each other and are Hermitian. Moreover, is S is compact then S^t is compact and hence, so is S^*. Hence $\Re(S) = (S + S^*)/2$ and $\Im(S) = (S - S^*)/2\sqrt{-1}$ are also compact. Hence, we want to consider the study $S = P + \sqrt{-1}Q$ where P and Q are compact Hermitian operators that commute with each other.

Suppose P and Q are commuting operators. Then, for an eigenvector v of P we have

$$PQv = QPv = Q\lambda v = \lambda Qv$$

where λ is the eigenvalue of P associated with the eigenvector v. Thus, if we denote by V_λ the space of all vectors v (including 0!) for which $Pv = \lambda v$, then Q takes this subspace to itself.

First we use the fact that P is compact Hermitian. Let V_n be the eigenspaces of P constructed in the previous subsection. As seen above, these are orthogonal and the direct sum of all of these is dense in H. (Recall that the direct sum only contains *finite* linear combinations.) Since each V_n is stable under Q as proved above, we can further apply the result to decompose each by using the fact that Q is compact Hermitian. Combining the orthonormal Hilbert bases for each V_n, we obtain

Exercise: Show that there is a collection $(e_\alpha)_{\alpha\in A}$ of unit vectors that form a Hilbert basis of H so that each e_α is an eigenvector for both P and Q. It follows that this is a Hilbert basis consisting of unit eigenvectors for S as well.

This shows that given a compact normal operator $S : H \to H$, there is a Hilbert basis $(e_\alpha)_{\alpha\in A}$ of H that consists of unit eigenvectors for S.