It is possible that some of you have given the right answer for the wrong reasons. It is possible that some of you have given the wrong answer for the right reasons. Both types should know that knowing the right reasons will prove more useful in the long run!

1. Let ℓ_1 denote the usual space of absolutely summable sequences of complex numbers. Consider the operator $R : \ell_1 \to \ell_1$ defined by (right shift)

$$R((a_1, a_2, \ldots)) = (0, a_1, a_2, \ldots)$$

Mark the following statements as True/False.

(1 mark) (a) 0 is an eigenvalue of R.

(a) **False**

(1 mark) (b) 0 is in the spectrum of R.

(b) **True**

(1 mark) (c) The operator R has no non-zero eigenvalues.

(c) **True**

(1 mark) (d) The spectrum of R is the singleton set $\{0\}$.

(d) **False**

Solution: One can show that the spectrum of R is the closed unit disk in the complex plane.

2. Consider the vector space $\mathbb{C}[T]$ of polynomials in one variable over the field of complex numbers with norms given by

$$\|P\|_C = \sup_{t\in[0,1]} |P(t)| \quad \text{and} \quad \|P\|_D = \sup_{t\in[0,1/2]} |P(t)|$$

Let $P_n(T) = T^n$. Mark the following statements as True/False.

(1 mark) (a) For each $t \in [0, 1]$ the sequence $P_n(t)$ of complex numbers converges in $\mathbb{C}$.

(a) **True**

(1 mark) (b) The sequence P_n is a Cauchy sequence on $\mathbb{C}[T]$ with respect to the norm $\|\cdot\|_C$.

(b) **False**

(1 mark) (c) The sequence P_n is a Cauchy sequence on $\mathbb{C}[T]$ with respect to the norm $\|\cdot\|_D$.

(c) **True**

(1 mark) (d) The linear functional $\mathbb{C}[T] \to \mathbb{C}$ defined by $P \mapsto \int_0^1 P(t)dt$ is a *continuous* linear functional with respect to the norm $\|\cdot\|_C$.

(d) **True**

(1 mark) (e) The sequence of values $\int_0^1 P_n(t)dt$ converges in $\mathbb{C}$.

(e) **True**

Solution: The sequence $P_n(t)$ converges pointwise to 0 for all $t \in [0, 1)$ and to 1 for $t = 1$. This limit is not a continuous function, hence the convergence is not in $\|\cdot\|_C$.

The integral is a bounded linear functional of norm 1.

The integrals are $1/(n + 1)$ which converge to 0.

3. Let ℓ_2 denote the space of square summable sequences of complex numbers with the usual *Hermitian* inner-product $\langle a, b\rangle = \sum_{n=1}^{\infty} a_n\overline{b_n}$. Mark the following statements as True/False.

(1 mark) (a) Given a linear functional $f : \ell_2 \to \mathbb{C}$, there is an element v_f in ℓ_2 so that for all w in ℓ_2 we have $f(w) = \langle w, v_f\rangle$.

(a) **False**

(1 mark) (b) The above statement (a) is *only* true if f is also continuous.

(b) **True**

(1 mark) (c) The above statement (a) is *only* true if f takes values in $\mathbb{R}$.

(c) **False**

(1 mark) (d) Given a linear functional $f : \ell_2 \to \mathbb{R}$ there is a linear functional $g : \ell_2 \to \mathbb{C}$ so that $f = \Re(g)$ (here $\Re : \mathbb{C} \to \mathbb{R}$ denotes the real part).

(d) **True**

(1 mark) (e) The above statement (d) is *only* true if f is also continuous.

(e) **False**

4. Let $\mathbb{C}^\infty$ be the space of all sequences $a = (a_1, a_2, \ldots, 0, \ldots,)$ of complex numbers that are eventually 0. Consider the usual norms

$$\|a\|_1 = \sum_{n=1}^{\infty} |a_n| \quad \text{and} \quad \|a\|_2 = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}$$

We study the linear transformation T from $\mathbb{C}^\infty$ to itself given by

$$T((a_1, a_2, \dots)) = (a_1, a_2/2, a_3/3, \dots)$$

Find the constants below *or* indicate that no such constant exists.

(1 mark) (a) Find a constant C_1 so that

$$\|a\|_2 \leq C_1 \|a\|_1$$

(a) $C_1 = 1$

Solution: Note that for any *finite* sequence $(a_1, \dots, a_N)$ we have

$$\left(\sum_{n=1}^{N} |a_n| \right)^2 = \sum_{n=1}^{N} \sum_{k=1}^{N} |a_n||a_k| \geq \sum_{n=1}^{N} |a_n|^2$$

It follows that $C_1 = 1$ will work.

(1 mark) (b) Find a constant C_2 so that

$$\|a\|_1 \leq C_2 \|a\|_2$$

(b) **None**

Solution: Consider $v^{(n)} = (1, 1/2, \dots, 1/n, 0, \dots)$. We know that $\|v^{(n)}\|_2 \leq \pi/\sqrt{6}$ for all n. However, $\|v^{(n)}\|$ goes to infinity as n goes to infinity. So there is no such constant C_2.

(1 mark) (c) Find a constant D_1 so that

$$\|T(a)\|_1 \leq D_1 \|a\|_1$$

(c) $D_1 = 1$

Solution: We note that

$$\sum_{n=1}^{N} |a_n/n| \leq \sum_{n=1}^{N} |a_n|$$

It follows that $D_1 = 1$ will work.

(1 mark) (d) Find a constant D_2 so that

$$\|T(a)\|_2 \leq D_2 \|a\|_2$$

(d) $D_2 = 1$

Solution: We note that

$$\sum_{n=1}^{N} |a_n/n|^2 \leq \sum_{n=1}^{N} |a_n|^2$$

It follows that $D_2 = 1$ will work.

(1 mark) (e) Find a constant D_3 so that

$$\|T(a)\|_1 \leq D_3 \|a\|_2$$

(**Warning**: Note the subscripts!)

(e) $D_3 = \pi/\sqrt{6}$

Solution: By the Cauchy-Schwarz inequality

$$\left(\sum_{n=1}^{N} |a_n/n| \right)^2 \leq \left(\sum_{n=1}^{N} |a_n|^2 \right) \cdot \left(\sum_{n=1}^{N} |1/n|^2 \right) \leq \frac{\pi^2}{6} \cdot \left(\sum_{n=1}^{N} |a_n|^2 \right)$$

So $D_3 = \pi/\sqrt{6}$ will work.

(1 mark) (f) Find a constant D_4 so that

$$\|T(a)\|_2 \leq D_4 \|a\|_1$$

(**Warning**: Note the subscripts!)

(f) $D_4 = 1$

Solution: By (a) above and by (b) above, we have

$$\|T(a)\|_2 \leq C_1 \|T(a)\|_1 \text{ and } \|T(a)\|_1 \leq D_1 \|a\|_1$$

It follows that if $D_4 = C_1 \cdot D_1$

$$\|T(a)\|_2 \leq D_4 \|a\|_1$$