## Spaces of Functions

The fundamental space of functions that we know is the space of polynomials. The vector space of (complex valued) polynomials in the variable $x$ can be identified with the space $\mathbb{C}^\infty$ via the map

$$a = (a_0, \dots, a_N, 0, \dots) \mapsto P_a(x) = \sum_{k=0}^{N} a_k x^k$$

(Note that for convenience of notation, we use sequences indexed starting from 0 in this section.)

We can consider this as a function $[0,1] \to \mathbb{C}$ by sending $t$ to $P_a(t)$. Let us define:

$$\|a\|_{C[0,1]} = \sup_{t\in[0,1]} |P_a(t)|$$

**Exercise**: Check that $\|a\|_{C[0,1]}$ is a norm on $\mathbb{C}^\infty$.

We will study the completion $V$ of $\mathbb{C}^\infty$ and show that it can be identified with the space of (complex valued) continuous functions on $C[0,1]$ (this explains the notation!). In order to do this, we must show:

1. Given $v$ in $V$ we can define a function $f_v : [0,1] \to \mathbb{C}$.
2. The map $v \mapsto f_v$ is one-to-one.
3. The function $f_v$ is continuous.
4. Given a continuous function $f : [0,1] \to \mathbb{C}$, there is a $v$ in $V$ so that $f = f_v$.

In addition, this result can, with suitable modifications be generalised to other compact subsets of $\mathbb{R}$. With some additional modifications, we can even generalise it to compact subsets of $\mathbb{R}^n$. *However*, it is worth pointing out that the case of polynomial functions on compact subsets of $\mathbb{C}$ is quite different! One important reason is that a polynomial function of two real variables $x$ and $y$ is quite different from a polynomial function of the *single* complex variable $z = x + \iota y$. In particular, $|z|^2 = x^2 + y^2$ *is* a polynomial function of $x$ and $y$ but is *not* a polynomial function of $z$. We will see, during the course of the proof, why this is important.

### Evaluation as a linear functional

Given $t$ in $[0,1]$, we have the linear functional $e_t : \mathbb{C}^\infty \to \mathbb{C}$ defined by

$$e_t(a) = P_a(t) = \sum_{k=0}^{\infty} a_k t^t$$

note that the sum on the right-hand side is *finite* since $a = (a_0, \dots, a_N, 0, \dots)$ is a sequence which consists of 0's beyond some index.

**Exercise**: Show that $e_t$ is a linear functional on $\mathbb{C}^\infty$ with norm $\|e_t\| \leq 1$ with respect to the norm $\|\cdot\|_{C[0,1]}$ on $\mathbb{C}^\infty$.

Since $V$ is the completion of $\mathbb{C}^\infty$ with respect to this norm, the continuous linear functional extends to a linear functional $e_t : V \to \mathbb{C}$ with norm 1. In particular, for any vector $v$ in $V$, and any $t$ in $[0,1]$, we have a complex number $e_t(v)$.

We now define the map $V \to \mathrm{Map}([0,1],\mathbb{C})$ given by $v \mapsto f_v$ where $f_v(t) = e_t(v)$. It is clear that $a$ in $\mathbb{C}^\infty$ is associated with the polynomial function $P_a$ by this assignment.

### $V$ as a space of functions

Given an element $v$ in $V$, we wish to show that, if $f_v(t) = 0$ for all $t$ in $[0,1]$, then $v$ is itself 0. Now, $v$ is determined by sequence $a^{(n)}$ of elements of $\mathbb{C}^\infty$ which converges to $v$ in the norm $\|\cdot\|_{C[0,1]}$. Thus, we would like to prove that for all $\epsilon > 0$, there is an $N$ so that $\|a^{(n)}\| < \epsilon$ for $n \geq N$. To ease the notation, we use $P_n$ to denote the polynomial function associated with $a^{(n)}$ as above; we also use the notation $\|\cdot\|$ to denote the norm $\|\cdot\|_{C[0,1]}$.

Since $a^{(n)}$ is a Cauchy sequence, there is a natural number $M_0$ so that $\|a^{(n)} - a^{(m)}\| < \epsilon/3$ for $n,m \geq M_0$. Applying the definition of this norm we see that $|P_n(y) - P_m(y)| < \epsilon/3$ for all $y$ in $[0,1]$.

Since $a^{(n)}$ converges to $v$ and for each $t$ in $[0,1]$, the map $e_t$ is a continuous linear functional on $V$ and $e_t(v) = 0$, there is an $N_t \geq M_0$ so that $|e_t(a^{(n)})| < \epsilon/3$ for all $n \geq N_t$. Equivalently, by definition of $e_t$, we have $|P_n(t)| < \epsilon/3$ for $n \geq N_t$.

Now $P_{N_t}$ is a polynomial function and thus it is continuous. It follows that there is a $\delta_t > 0$ so that $|P_{N_t}(y) - P_{N_t}(t)| < \epsilon/3$ for all $y$ in the interval $(t - \delta_t, t + \delta_t)$.

**Exercise**: Show that for all $n \geq N_t$, and for all $y$ in the interval $(t - \delta_t, t + \delta_t)$ we have $\|P_n(y)\| < \epsilon$. (Hint: Combine the three inequalities using the triangle inequality!)

Since $[0,1]$ is a compact set, there is a finite collection $t_1, \ldots, t_r$ of points in $[0,1]$ so that the union of the intervals $(t_i - \delta_{t_i}, t_i + \delta_{t_i})$ cover the entire interval $[0,1]$. Now, put $N = \max_{i=1}^r N_{t_i}$.

**Exercise**: Show that for all $n \geq N$, and for all $y$ in the interval $[0,1]$ we have $\|P_n(y)\| < \epsilon$. (Hint: For each $y$ choose an appropriate $t_i$ and apply the previous exercise.)

It follows that $\|a^{(n)}\| < \epsilon$ for all $n \geq N$ as required.

### Continuity of $f_v$

We want to prove that given $t_0$ in $[0,1]$ and $\epsilon > 0$, there is a $\delta > 0$ so that $|f_v(t) - f_v(t_0)| \leq \epsilon$ for all $t$ in $(t_0 - \delta, t_0 + \delta)$.

As in the previous paragraph, let $a^{(n)}$ be a sequence of elements of $\mathbb{C}^\infty$ which converges to $v$ in the norm $\|\cdot\|_{C[0,1]}$; to simplify notation we use $P_n$ to denote the polynomial associated with $a^{(n)}$. Continuing as before, let $M_0$ be such that that $\|a^{(n)} - a^{(m)}\| < \epsilon/3$ for $n, m \geq M_0$. Applying the definition of this norm we see that $|P_n(y) - P_m(y)| < \epsilon/3$ for all $y$ in $[0,1]$.

Since $e_{t_0}$ is a continuous linear functional, there is an $N \geq M_0$ so that $|e_{t_0}(v) - e_{t_0}(a^{(n)})| < \epsilon/3$ for all $n \geq N$. We have $e_{t_0}(a^{(n)}) = P_n(t_0)$ and $e_{t_0}(v) = f_v(t_0)$.

Since $P_N$ is a continuous function of $t$, there is a $\delta > 0$ so that $|P_N(t) - P_N(t_0)| < \epsilon/3$ for all $t$ in $(t_0 - \delta, t_0 + \delta)$.

**Exercise**: Combine the above inequalities to show that $|P_n(t) - f_v(t_0)| < \epsilon$ for all $n \geq N$ and $t$ in $(t_0 - \delta, t_0 + \delta)$.

By the continuity of $e_t$, we see that $f_v(t) = e_t(v)$ is the limit of $P_n(t) = e_t(a^{(n)})$ as $n$ goes to infinity. The limit of the above inequalities gives us $|f_v(t) - f_v(t_0)| \leq \epsilon$ for all $t$ in $(t_0 - \delta, t_0 + \delta)$ as required.

### Real and Imaginary parts

An element $a$ in $\mathbb{C}^\infty$ can be separated into real and imaginary parts by writing $a = u + \sqrt{-1}v$ where $u$ and $v$ lie in $\mathbb{R}^\infty$. Moreover, since $t$ in $[0,1]$ is a real number $P_u(t)$ is the real part of $P_a(t)$ and $P_v(t)$ is the imaginary part of $P_a(t)$. Thus, if we prove that the completion $V_{\mathbb{R}}$ of $\mathbb{R}^\infty$ has image equal to the *real* valued continuous functions on $[0,1]$, then it can be deduced that the image $V$ is equal to the complex valued continuous functions on $[0,1]$. Thus, we will now prove the statement for real continuous functions.

The reason that this is a useful reduction is as follows. For a real valued function $f$, we define $f_+ = \max\{f, 0\}$ and $f_- = -\min\{f, 0\}$; these are non-negative functions. Moreover, $f = f_+ - f_-$ and $|f| = f_+ + f_-$. It follows that $f_+ = (f + |f|)/2$. Now, if $f$ and $g$ are two functions, then $\max\{f, g\} = \max\{f - g, 0\} + g$ and $\min\{f, g\} = -\max\{-f, -g\}$. Thus, for real-valued functions, if we want to exhibit the latter two functions in a certain vector space of functions, it is enough if we should that for every $f$ in that vector space, the function $|f|$ is in that vector space of functions.

Secondly, for real-valued functions, $|f| = \sqrt{f^2}$. So, what we really need to show is that if $f$ is in the vector space then so is $\sqrt{f^2}$.

**Polynomials of polynomials**

**Exercise**: If $P$ is a polynomial function of $n$ variables and $Q_1, \ldots, Q_n$ are polynomial functions of $m$ variables then $P(Q_1, \ldots, Q_n)$ is a polynomial function of $m$ variables.

This can be used to show:

**Exercise**: If $f$ lies in the image of $V_{\mathbb{R}}$ and $P$ is a real polynomial function of one variable, then $P(f)$ lies in the image of $V_{\mathbb{R}}$.

Now, suppose that $P_n$ is a sequence of polynomials converging uniformly in $[0,1]$ to a function $g$ and $f$ takes values in $[0,1]$.

**Exercise**: Show that $P_n(f)$ converges uniformly to $g(f)$.

It follows that if $g$ lies in $V_{\mathbb{R}}$ and $f$ takes values in $[0,1]$, then $g(f)$ is also in $V_{\mathbb{R}}$. This will allow us to apply the following construction.

**Square roots**

We now produce a sequence of polynomials which converge to the function $s(t) = \sqrt{t}$ for $t$ in $[0,1]$.

Let $u_1(t) = t$ and we inductively define for $n \geq 1$:

$$u_{n+1}(t) = u_n(t) + \frac{t - (u_n(t))^2}{2}$$

**Exercise**: Check by induction that if $P$ is a polynomial then so is $P + (t - P^2)/2$.

**Exercise**: Suppose that if $0 \leq a \leq \sqrt{t} \leq 1$, then $(t - a^2)/2 \leq \sqrt{t} - a$. (Hint: Note that $\sqrt{t} + a \leq 2$.)

**Exercise**: Check by induction that $0 \leq u_{n+1}(t) \leq \sqrt{t}$ for $t$ in $[0,1]$.

Thus, $u_{n+1}$ is an increasing sequence of polynomials bounded above by $s(t)$. Let $u(t) = \sup_n u_n(t)$.

**Exercise**: Check that $t = u(t)^2$. (Hint: Take limits in the above equation.)

In other words, we have shown that $s(t)$ is the pointwise limit of the polynomials $u_n(t)$ for $t$ in $[0,1]$. We want to show that this convergence is in norm. In other words, given $\epsilon > 0$ we want to find $N$ so that $|u_n(t) - s(t)| < \epsilon$ for all $n \geq N$ and for all $t$ in $[0,1]$. This is a consequence of the following three features of this situation:

- The functions $s(t)$ and $u_n(t)$ are continuous.
- The values $u_n(t)$ increase to $s(t)$ as $n$ increases.
- The interval $[0,1]$ is compact.

The idea is to find for each point $t$ in $[0,1]$

- A $\mu_t$ for which $|s(y) - s(t)| < \epsilon/3$ if $y$ lies in $(t - \mu_t, t + \mu_t)$.
- An $N_t$ for which $u_{N_t}(t) > s(t) - \epsilon/3$.
- A $\tau_t$ for which $|u_{N_t}(y) - u_{N_t}(t)| < \epsilon/3$ if $y$ lies in $(t - \tau_t, t + \tau + t)$.

We then take $\delta_t$ to be the minimum of $\mu_t$ and $\tau_t$. Now, we use compactness of $[0,1]$ to find a finite collection $t_1, \dots, t_r$ so that the union of $(t_i - \delta_{t_i}, t_i + \delta_{t_i})$ covert $[0,1]$. Let $N$ be greater than or equal to all the $N_{t_i}$.

**Exercise**: Combine the above inequalities to check that $|u_n(y) - s(y)| < \epsilon$ for all $y$ in $[0,1]$ and for all $n \geq N$.

This shows that $s(t) = \sqrt{t}$ is a uniform limit of polynomials in $[0,1]$.

Given that a certain function $f$ lies in $V_{\mathbb{R}}$ we want to show that $|f|$ lies in $V_{\mathbb{R}}$. Suppose that $a = \|f\|$ is the supremum of all values of $f$. Then $f^2/a^2$ takes values in $[0,1]$. Combining the results of the previous two subsections, we can then show that $\sqrt{f^2/a^2}$ is in $V_{\mathbb{R}}$. This means that $|f| = a\sqrt{f^2/a^2}$ is in $V_{\mathbb{R}}$. As seen in two subsections above, this means that if $f$ and $g$ are in $V_{\mathbb{R}}$ then so are $\min\{f, g\}$ and $\max\{f, g\}$.

### Min-Max approach

Given a continuous function $f : [0,1] \to \mathbb{R}$ we want to approximate it by functions in $V_{\mathbb{R}}$. More precisely, given $\epsilon > 0$ we want to find a function $g$ in $V_{\mathbb{R}}$ so that $|f(t) - g(t)| < \epsilon$ for all $t$ in $[0,1]$. We think of this condition as

$$f(t) - \epsilon < g(t) < f(t) + \epsilon$$

and try to satisfy each side "separately".

**Exercise**: For every chosen pair of points $t$, $s$ in $[0,1]$ we can find a polynomial $Q_{t,s}$ so that $Q_{t,s}(t) = f(t)$ and $Q_{t,s}(s) = f(s)$. (Hint: If $t \neq s$, consider $Q_{t,s} = f(s)(x-t)/(s-t) + f(t)(x-s)/(t-s)$. What about if $s = t$?)

Now $f$ and $Q_{t,s}$ are continuous on $[0,1]$ and $f(s) = Q_{t,s}(s)$. So, there is a $\delta_{t,s}$ so that $|f(a) - Q_{t,s}(a)| < \epsilon$ for each $a$ in $[0,1]$ that lies in $(s - \delta_{t,s}, s + \delta_{t,s})$. Using compactness of $[0,1]$, there is a finite collection $s_1, \dots, s_m$ so that if $\delta_i = \delta_{t,s_i}$ then $[0,1]$ is contained in the union of the intervals $(s_i - \delta_i, s_i + \delta_i)$. We put $h_t = \min_{i=1}^m Q_{t,s_i}$. As seen above $h_t$ lies in $V_{\mathbb{R}}$.

**Exercise**: Check that $h_t(t) = f(t)$ and that $h_t(a) < f(a) + \epsilon$ for all $a$ in $[0,1]$. (Hint: $h_t$ is the minimum of functions $Q_{t,s_i}$ and *at least one* of these satisfies this condition at each point of $[0,1]$.)

Now, $h_t$ and $f$ are continuous in $[0,1]$ and $h_t(t) = f(t)$. So, there is a $\mu_t$ so that $|f(a) - h_t(a)| < \epsilon$ for each $a$ in $[0,1]$ which lines in $(t - \mu_t, t + \mu + t)$. Again using the compactness of $[0,1]$, there is a finite collection $t_1, \dots, t_n$ so that if

$\mu_i = \mu_{t_i}$, then $[0,1]$ is contained in the union of the intervals $(t_i - \mu_i, t_i + \mu + i)$. We put $g = \max_{i=1}^m h_{t_i}$. As seen above $g$ lies in $V_{\mathbb{R}}$.

**Exercise**: Check that $f(a) - \epsilon < g(a) < f(a) + \epsilon$ for all $a$ in $[0,1]$. (Hint: The second part of the inequality is satisfied by all the $h_{t_i}$. For the first part, we note that $g$ is the maximum of functions $h_{t_i}$ and *at least one* of these satisfies this condition at each point of $[0,1]$.)

Thus, we have completed the argument that $V_{\mathbb{R}}$ contains all continuous functions on $[0,1]$ with values in $\mathbb{R}$. As seen above, the argument for $V$ can be completed by arguing separately for the real and imaginary parts.

### Extensions

The arguments given above can be extended to any compact set $K$ in $\mathbb{R}$ by defining

$$\|a\|_{C(K)} = \sup_{x \in K} P_a(x)$$

as a norm on $\mathbb{C}^\infty$. We will still need to use the subsection above to find the square root on the interval $[0,1]$ as a uniform limit of polynomials on $[0,1]$. The subsection on the composition of polynomials then allows us to show that the completion is closed under $f \mapsto |f|$. The rest of the proof (Min-Max part) is the same.

In order to extend the argument to compact sets in $\mathbb{R}^n$, we need to put an *order* on monomials $t_1^{k_1} \cdots t_n^{k_n}$ in order to identify $\mathbb{C}^\infty$ with $\mathbb{C}[t_1, \ldots, t_n]$. Other than that the proof will proceed along similar lines.

**Warning**: One *can* identify $\mathbb{C}$ with $\mathbb{R}^2$. However, polynomials with complex coefficients on $\mathbb{R}^2$ are actually polynomials in 2 variables. To think of them as polynomials in the variable $z \in \mathbb{C}$ we need to write $x = (z + \overline{z})/2$ and $y = -\iota(z - \overline{z})/2$. With this substitution, these become polynomials with complex coefficients in two variables $z$ and $\overline{z}$. On the other hand, if we take the norm

$$\|a\| = \sup_{|z| \leq 1} \left| \sum_{i=0}^{n} a_i z^i \right|$$

(where $z$ is allowed to take complex values) on $\mathbb{C}^\infty$, then the completion is quite a bit *smaller* than the space of continuous functions on the unit disk. By Montel's theorem one can show that this completion consists of those continuous functions on the unit disk that are *analytic* in the interior of the unit disk.