Invertible operators and spectrum

An bounded linear operator $T : V \to V$ from a normed linear space to itself is called "invertible" if there is a bounded linear operator $S : V \to V$ so that $S \circ T$ and $T \circ S$ are the identity operator $\mathbf{1}$. We say that S is the inverse of T in this case.

Exercise: If S and T are bounded linear operators that are inverses of each other, show that $\|S\|\|T\| \leq 1$.

We note that for a linear operator to have an inverse (*without* the condition of boundedness) it is necessary and sufficient that the operator be one-to-one and onto. In general, this may not be enough to ensure that this inverse operator be bounded.

For the arguments in this section to work, we will need to talk about convergence of series and thus from here on we will restrict our attention to *complete* normed linear spaces (i.e. Banach spaces). Note that in this case the space $B(V)$ of bounded linear operators from V to itself is a Banach space with respect to the operator norm.

Series in a Banach spaces

Let x_n be a sequence of elements in a Banach space V and define $s_N = \sum_{n=1}^{N} x_n$. We say that the series $\sum_{n=1}^{\infty} x_n$ *converges* in V if s_N is a Cauchy sequence in V. In that case, s_N converges to a unique s in V and we say that s is the sum of this series.

The most important case when we have a convergent series is when the sum $\sum_{n=1}^{N} \|x_n\|$ is uniformly bounded independent of N. In this case, we say that the series is *absolutely* convergent.

Exercise: For an absolutely convergent series show that, for all $\epsilon > 0$ there is an N so that $\sum_{n=N+1}^{M} \|x_n\| < \epsilon$ for *all* $M > N$. (Hint: Use proof by contradiction!)

By the triangle inequality, one can then show:

Exercise: For all $\epsilon > 0$, and for N as above, show that $\|s_n - s_m\| < \epsilon$ for $n, m > N$. (Hint: Expand $s_n - s_m$ as a sum of x_k's.)

In other words, s_n is a Cauchy sequence. We thus see that an absolutely convergent series is also convergent.

Exercise: If $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are convergent series in V, then

$$\sum_{n=1}^{\infty} (x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$$

(Hint: The proof is the same as that for series of real numbers with absolute value replaced by norm.)

Neighbourhood of identity

Let $T : V \to V$ be a bounded linear operator such that $\|T\| < 1$ for a Banach space V. We consider the series $\sum_{n=0}^{\infty} T^n$ in the Banach space $B(V)$. (Here and below we will use the convention that $T^0 = \mathbf{1}$ is the identity operator. This simplifies the notation.)

Exercise: Show that the series above is absolutely convergent. (Hint: Pick an $k > 0$ so that $\|T\| < (1 - 1/k)$ and show that $\|T^n\| < (1 - 1/k)^n$. Now use comparison to prove absolute convergence.)

Since $B(V)$ is a Banach space, this converges to an S in $B(V)$.

Exercise: For each v in V check that the series $\sum_{n=0}^{\infty} T^n(v)$ is absolutely convergent in V. (Hint: Note that $\|T^n(v)\| \leq \|T^n\|\|v\|$.)

Exercise: Check that $S(v) = \sum_{n=0}^{\infty} T^n(v)$. (Hint: Let S_N be the partial sums $\sum_{n=0}^{N} T^n$ and note that $S_N(v) = \sum_{n=0}^{N} T^n(v)$.)

Exercise: Check that $T(S(v)) = v + S(v)$ and $S(T(v)) = v + S(v)$ for all v in V.

We can now conclude the following.

Exercise: Given that A is an element of $B(V)$ such that $\|A - \mathbf{1}\| < 1$; i.e. that A at a distance less than 1 from the identity operator $\mathbf{1} : V \to V$. Show that A is invertible. (Hint: Put $T = \mathbf{1} - A$ and apply the previous exercises.)

In other words, *all* operators at a distance of less than 1 from the identity operator are invertible.

We can generalise this to neighbourhoods of invertible operators other than $\mathbf{1}$. Given an invertible operator P, let Q be its inverse.

Exercise: Given that $\|A - P\| < 1/\|Q\|$, show that A is invertible. (Hint: Note that if $T = \mathbf{1} - Q \circ A = Q \circ (P - A)$, then $\|T\| < 1$. Now use this to show that $Q \circ A$ is invertible and thus A is invertible.)

If we use $GL(V)$ to denote the subset of $B(V)$ that consists of invertible elements, then this shows that $GL(V)$ is *open*.

Spectrum of an operator

For an operator A in $B(V)$, the spectrum is defined as the set of $z \in \mathbf{C}$ such that $A - z\mathbf{1}$ is *not* invertible. The set $\sigma(A)$ is used to denote this set.

If v is a non-zero vector and λ a complex number such that $Av = \lambda v$ then we say that v is an eigenvector of A and λ is an eigenvalue of A. It is clear that $A - \lambda\mathbf{1}$ is not one-to-one in this case, so it is not invertible and thus λ lies in $\sigma(A)$. However, we shall see examples where the spectrum *can be larger* than the set of eigenvalues of an operator.

If $z \neq 0$, then this condition is the same as the condition that $\mathbf{1} - (1/z)A$ is not invertible. As seen in the previous section, if $\|(1/z)A\| < 1$, then $\mathbf{1} - (1/z)A$ *is* invertible. So, we see that if $|z| > \|A\|$, the operator $A - z\mathbf{1}$ is invertible. Thus $\sigma(A)$ is contained in the closed disk of radius $\|A\|$ around the origin in $\mathbf{C}$.

The map $\phi : z \mapsto A - z\mathbf{1}$ is a continuous map $\mathbf{C} \to B(V)$. As seen in the previous section $GL(V)$ is open in $B(V)$. Hence, its complement is closed in $B(V)$. The spectrum $\sigma(A)$ is *precisely* $\phi^{-1}(B(V) - GL(V))$, so it is a closed set.

In other words $\sigma(A)$ is a closed and bounded subset of $\mathbf{C}$; i.e. it is a compact set.

The complement $\mathbf{C} - \sigma(A)$ is called the *resolvent* set of A and is sometimes denoted by $\rho(A)$. It is a non-empty (as seen above) open subset of $\mathbf{C}$.

Matrix Coefficients of an operator

Given a linear operator $A : V \to V$, f a bounded linear functional on V and v a vector in V, we can think of $f(Av)$ as a "matrix coefficient" of A. In the traditional finite dimensional case, suppose e_i for $i = 1, \ldots, n$ is a basis of V and e_i^* for $i = 1, \ldots, n$ is a dual basis. Then

Exercise: Check that $e_i^*(Ae_j)$ is the (i, j)-th entry of the matrix of A with respect to the basis with notation as above.

Exercise: Given a bounded linear operator A such that $f(Av) = 0$ for *every* vector v and *every* bounded linear functional f on V, show that $A = 0$. (Hint: If $Av \neq 0$ for some v, then use Hahn-Banach theorem to show that there is an f so that $f(Av) \neq 0$.)

Thus, a bounded linear operator is determined by its matrix coefficients. For an operator A, consider the function $z \mapsto f((A - z\mathbf{1})^{-1}(v))$ defined in the resolvent set $\rho(A)$. This is a complex valued function on an open subset of the complex plane. We now study this function.

Exercise: Show that $\|A - z\mathbf{1}\| \geq |z| - \|A\|$. (Hint: Use the triangle inequality for operator norm and the fact that the norm of $\mathbf{1}$ is 1.)

We have seen that if $|z| > \|A\|$, then $A - z\mathbf{1}$ is invertible. Moreover, we have seen that

$$\|(A - z\mathbf{1})^{-1}\| \leq \frac{1}{\|A - z\mathbf{1}\|}$$

Exercise: Show that $f((A - z\mathbf{1})^{-1}(v))$ is a bounded function on the subset of $\rho(A)$ that consists of z such that $|z| > \|A\|$.

Exercise: Suppose that B is an invertible operator. Show that $\sum_{n=1}^{\infty} z^n f(B^{-n}Bv)$ is an absolutely convergent series for z such that $|z| < 1/\|B^{-1}\|$.

It follows that this series defines an analytic function of z in this disk.

Exercise: Check that $f((B - z\mathbf{1})^{-1}(v))$ is given by the above infinite series for $|z| < 1/\|B^{-1}\|$.

Exercise: Apply the above exercise to $B = A - z_0\mathbf{1}$ for some z_0 in $\rho(A)$, to show that $f((A - z\mathbf{1})^{-1}(v))$ is an analytic function of z in some disk around z_0. (Hint: Write $A - z\mathbf{1}$ as $B - (z - z_0)\mathbf{1}$.)

It follows that $f((A-z\mathbf{1})^{-1}(v))$ is an analytic function for z in the open set $\rho(A)$. We have also seen that it is bounded outside the disk of radius $\|A\|$. Hence, if $\rho(A) = \mathbf{C}$, the this is a bounded analytic function in the entire plane. Such a function is necessarily a constant independent of z.

In other words, if $\sigma(A)$ were to be empty, then all the matrix coefficients of $(A - z\mathbf{1})^{-1}$ would be constant (and the same as the matrix coefficients of A^{-1}). By what we have seen above, this would mean that $(A - z\mathbf{1})^{-1}$ would itself be the constant operator; consequently the same would be true of $A - z\mathbf{1}$. However, this would mean $A = A - \mathbf{1}$ or $1 = 0$; a contradiction. Hence, we have proved by contradiction that $\sigma(A)$ is non-empty.

In conclusion, the spectrum $\sigma(A)$ of a bounded linear operator A on a Banach space is a *non-empty compact* subset of $\mathbf{C}$.

Diagonal operators

Let V be one of the spaces $\mathcal{C}_0$, $\mathcal{C}$, ℓ_1 or ℓ_2 and let $e^{(n)}$ for $n = 1, \ldots,$ be the standard basis.

Exercise: Given any bounded sequence (a_n) of complex numbers, define the operator T by $T((b_n)) = (a_n b_n)$. Show that this is a bounded operator on V and that $\|T\| \leq \|(a_n)\|_\infty$.

If $\|a_n\| \geq \delta > 0$ for *all* n, then $(1/a_n)$ is also a bounded sequence with $\|(1/a_n)\|_\infty \leq 1/\delta$.

Exercise: If $S((b_n)) = (b_n/a_n)$, then show that S is the inverse of T.

Now, $Te^{(n)} = a_n e^{(n)}$ for all n. This shows that a_n is an eigenvalue of T and so lies in the spectrum $\sigma(T)$. Hence, $\sigma(T)$ contains the closure of the set $\{a_n : n \in \mathbf{N}\}$ of complex numbers.

Exercise: If T is given as above and a is any complex number, then check that $T - a\mathbf{1}$ is given by $(T - a\mathbf{1})((b_n)) = ((a_n - a)b_n)$.

Suppose a is *not* in the closure of this set. This means that there is a $\delta > 0$ so that $|a - a_n| \geq \delta$. Let $S((b_n)) = (b_n/(a_n - a)$. We have seen above that this is the inverse of the operator $T - a\mathbf{1}$. In other words, a is *not* in the spectrum $\sigma(T)$. Thus, the spectrum of T is *exactly* the closure of the set $\{a_N : n \in \mathbf{N}\}$.

Exercise: Given any non-empty compact set K in $\mathbf{C}$ show that there is a bounded sequence (a_n) of complex numbers so that the closure of $\{a_n : n \in \mathbf{N}\}$

is K.

Consider the bounded sequence $(1/n)$ and the associated operator T.

Exercise: Show that the operator $T((b_n)) = (b_n/n)$ is one-to-one.

It follows that 0 is *not* an eigenvalue of T. On the other hand, as seen above, 0 lies in the spectrum of T.