Write your name and/or registration number in the box provided.
Write your answers in space provided.
You have 1 hour to complete this exam.

Name:________________ *Reg. No:*________________

Question:	1	2	3	Total
Points:	13	4	3	20
Score:				

1. As usual, the space $\mathcal{C}$ consists of all Cauchy sequences of complex numbers with norm of a sequence (z_n) given by

$$\|(z_n)\|_\infty = \sup_{n=1}^{\infty} |z_n|$$

Define the elements $e^{(N)}$ in $\mathcal{C}$ for $N \geq 1$.

$$e_n^{(N)} = \begin{cases} 0 & \text{if } n \neq N \\ 1 & \text{if } n = N \end{cases}$$

Also define $e^{(0)}$ as the element $(1, 1, \dots)$ of $\mathcal{C}$ all of whose entries are 1.

(1 mark) (a) Consider the map $L : \mathbb{C}^\infty \to \mathcal{C}$ given by

$$(a_1, \dots, a_N, 0, 0, \dots) \mapsto \sum_{n=\mathbf{1}}^{N} a_n e^{(n)}$$

Is the map L one-to-one.

(a) **Yes; $e^{(n)}$ are linearly independent.**

(2 marks) (b) Is the image of L dense?

(b) **No; The closure of the image is $\mathcal{C}_0$**

(1 mark) (c) What is the norm of $L((a_n))$ for a general element (a_n) of $\mathbb{C}^\infty$ as given above? Denote this norm as $\|\cdot\|_L$.

Solution:

$$\|(a_n)\| = \sup_{n=1}^{\infty} |a_n|$$

(1 mark) (d) Consider the map $M : \mathbb{C}^\infty \to \mathcal{C}$ given by

$$(a_1, \dots, a_N, 0, 0, \dots) \mapsto a_1 e^{(0)} + \sum_{n=1}^{N} a_{n+1} e^{(n)}$$

Is the map M one-to-one.

(d) **Yes; $e^{(n)}$ are linearly independent.**

(2 marks) (e) Is the image of M dense?

(e) **Yes; $\mathcal{C} = \mathcal{C}_0 + \mathbb{C}e^{(0)}$**

(1 mark) (f) What is the norm of $M((a_n))$ for a general element (a_n) of $\mathbb{C}^\infty$ as given above? Denote this norm as $\|\cdot\|_M$.

Solution:

$$\|(a_n)\| = \sup_{n=1}^{\infty} |a_1 + a_{n+1}|$$

(2 marks) (g) Determine a constant C so that

$$\sup\{\|(a_n)\|_M : \|(a_n)\|_L = 1\} \le C$$

Solution: If $\sup_{n=1}^{\infty} |a_n| \le 1$, then

$$|a_1 + a_{n+1}| \le |a_1| + |a_{n+1}| \le 2$$

Hence $C = 2$ will work.

(2 marks) (h) Determine a constant D so that

$$\sup\{\|(a_n)\|_L : \|(a_n)\|_M = 1\} \le D$$

Solution: Note that for each (a_n) in $\mathbb{C}^\infty$, there is an N so that $a_n = 0$ for $n > N$. So if $\sup_{n=1}^{\infty} |a_1 + a_{n+1}| \le 1$, then $|a_1| \le 1$. It follows that

$$|a_{n+1}| \le |-a_1| + |a_1 + a_{n+1}| \le 2$$

Hence $D = 2$ will work.

(1 mark) (i) What is the relation between the two topologies on $\mathbb{C}^\infty$ induced by the two norms?

(i) **Topologies are the same since norms are equivalent**

2. As usual, the space ℓ_1 consists of all sequences (z_n) of complex numbers such that $\sum_{n=1}^{\infty}|z_n| < \infty$ on which we have the norm

$$\|(z_n)\|_1 = \sum_{n=1}^{\infty}|z_n|$$

Also, the space $\mathcal{C}$ consists of all Cauchy sequences of complex numbers with norm of a sequence (z_n) given by

$$\|(z_n)\|_\infty = \sup_{n=1}^{\infty}|z_n|$$

We define the map $S : \ell_1 \to \mathcal{C}$ by

$$(z_n) \mapsto (w_n = \sum_{k=1}^{n} z_k)$$

Mark each of the following statements as True or False.

(1 mark) (a) This map is one-to-one.

(a) **True;** $w_1 = z_1$**,** $w_2 - w_1 = z_1$ **and so on.**

(1 mark) (b) This map is continuous.

(b) **True; In fact its norm is clearly** ≤ 1

(1 mark) (c) This map preserves the norm.

(c) **False;** $a = (1/2, -1/2, 0, \dots)$ **maps to** $b = (1/2, 0, \dots)$ **and** $\|a\|_1 = 1$ **but** $\|b\|_\infty = 1/2$**.**

(1 mark) (d) This map is onto.

(d) **False;** $(1, 1 - 1/2, 1 - 1/2 + 1/3, \dots)$ **is not in the image.**

3. Given an inner product space V over $\mathbb{R}$ and vector u in V such that $\|u\| = 1$ in the induced norm.

(1 mark) (a) Is the following map a continuous linear functional on V?

$$w \mapsto \langle w, u \rangle$$

(a) **Yes;** $|\langle w, u \rangle| \leq \|w\|$

(1 mark) (b) Is the following map a norm-preserving map from V to V?

$$w \mapsto w - \langle w, u \rangle u$$

(b) **No;** u **maps to 0.**

(1 mark) (c) Is the following map a continuous *onto* map from V to V?

$$w \mapsto w - \mathbf{2}\langle w, u \rangle u$$

(c) **Yes; It is reflection in plane perpendicular to** u