Operator Spaces

Given normed linear spaces V and W, we have the space $B(V, W)$ of continuous (bounded) linear transformations from $V \to W$. We have seen that $B(V, W)$ is a linear space. Moreover, for L in $B(V, W)$, if we define:

$$\|L\| = \sup_{\|v\|=1} \|L(v)\|$$

then, we have shown that:

1. For L and M in $B(V, W)$, we have $\|L + M\| \leq \|L\| + \|M\|$.
2. For $z \in \mathbb{C}$ we have $\|z \cdot L\| = |z|\|L\|$.
3. If $\|L\| = 0$ then $L = 0$.

So, we can construct new normed spaces out of old ones!

Weak and Strong Convergence

Given a sequence L_n of elements in $B(V, W)$ and a map $L : V \to W$, we say that L_n converges *weakly* to L if, for every v in V, the sequence $L_n(v)$ of elements of W converges to $L(v)$. This is called "weak" as opposed to the stronger notion of convergence in the norm topology on $B(V, W)$.

Now $L_n(a \cdot v + w) = a \cdot L_n(v) + L_n(w)$. The left-hand side of this equation converges to $L(a \cdot v + w)$ by weak convergence. On the other hand, by continuity of scalar multiplication and addition of vectors in W, we see that

$$\lim_{n\to\infty} (a \cdot L_n(v) + L_n(w)) = a \cdot \lim_{n\to\infty} L_n(v) + \lim_{n\to\infty} L_n(w) = a \cdot L(v) + L(w)$$

Thus, it is *consequence* of weak convergence that the map $L : V \to W$ is linear! In other words:

The weak limit of a sequence of linear operators is linear.

Secondly, suppose that $\|L_n\| \leq C$ is uniformly bounded independent of n. We then have $\|L_n(v)\| \leq C\|v\|$. Since norm is a continuous function on W, the left-hand side of the inequality converges to $\|L(v)\|$. It follows that $\|L\| \leq C$ and thus $L : V \to W$ is bounded and hence continuous. In other words:

If a uniformly bounded sequence of linear operators weakly converges, then the limit is a bounded linear operator.

Now, in addition to L_n converging weakly to L, suppose that L_n is a Cauchy sequence in the operator norm. This means that for all $\epsilon > 0$, there is an N (depending on ϵ) such that $\|L_n - L_m\| < \epsilon/2$ for all $n, m \geq N$. We then have:

$$\|L_n(v) - L_N(v)\| < (\epsilon/2)\|v\| \text{ for all } n \geq N$$

Taking a limit of the left-hand side as n goes to infinity, we get:

$$\|L(v) - L_N(v)\| \leq (\epsilon/2)\|v\|$$

Applying the triangle inequality, we get, for $n \geq N$

$$\|L(v) - L_n(v)\| \leq \|L(v) - L_N(v)\| + \|L_n(v) - L_N(v)\| < (\epsilon/2)\|v\| + (\epsilon/2)\|v\| = \epsilon\|v\|$$

In other words, $\|L - L_n\| < \epsilon$ for $n \geq N$. Since we get such an N for every $\epsilon > 0$, it follows that L_n converges to L in the operator norm. In other words:

If a Cauchy sequence of linear operators converges weakly to a linear operator, then it also converges strongly to that linear operator.

Completeness

One particular case when we can apply the previous section is when W is complete. In that case, *if* $L_n(v)$ is a Cauchy sequence, then it converges to a vector in W. We then *define* $L(v)$ to be the limit.

For example, if we assume that L_n is a Cauchy sequence in the operator norm, then for each v, we can show that $L_n(v)$ is a Cauchy sequence as follows. First of all, this is clear when $v = 0$ since $L_n(0) = 0$ for all n. Thus we can assume that $\|v\| \neq 0$. Now, given $\epsilon > 0$, we know that there is an N (depending on ϵ and v) such that for all $n, m \geq N$, we have $\|L_n - L_m\| < \epsilon/\|v\|$. It follows that $\|L_n(v) - L_m(v)\| < \epsilon$ for all $n, m \geq N$. Since we can do this for every ϵ, it follows that $L_n(v)$ is Cauchy.

In other words, if W is complete, and if a sequence of elements L_n in $B(V, W)$ is a Cauchy sequence in the operator norm, then it weakly converges to a map $L : V \to W$. Now, as seen above, this means that L is a linear operator and L_n strongly converges to L. Since a Cauchy sequence is uniformly bounded, it follows that L is in $B(V, W)$. In summary, we have shown that:

If W is a complete normed space, then $B(V, W)$ is a complete normed space.

The above is specifically useful when applied to $B(V, \mathbb{C})$. Since $\mathbb{C}$ is a complete metric space with the usual norm on it, we see that the dual of a normed space is *always* complete.

This provides us with another proof that ℓ_1 which is the dual of $\mathcal{C}_0$ is complete. Similarly, ℓ_∞ is the dual of ℓ_1 and so it is complete. We will see further applications when we study Hilbert spaces and the spaces ℓ_p.