Solutions

(Questions include corrections and clarifications given during the examination.)

1. The casino has a game to keep rolling a fair die and stop once we have seen 1 three times. The player gets 1 rupee for each non-1 rolled.

(1 mark) (a) Write an *expression* for the precise probability that the player gets at most (less than or equal to) 15 rupees.

> **Solution:** Let W denote the random variable denoting the amount of money won. This follows the negative binomial distribution
>
> $$P(W = k) = \binom{k+3-1}{k} \cdot \left(1 - \frac{5}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^k$$
>
> Thus, the answer is
>
> $$\sum_{k=0}^{15} P(W = k) = \sum_{k=0}^{15} \binom{k+2}{k} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^k$$

(1 mark) (b) Write the *expression* for the expected winnings of the player

> **Solution:** This is
>
> $$\sum_{k=0}^{\infty} k \cdot \binom{k+3-1}{k} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^k$$

(1 mark) (c) Calculate the expected winnings of the player.

> **Solution:** As seen in class (or directly using the expectation for Negative Binomial distribution) this is
>
> $$\frac{(5/6) \cdot 3}{(1/6)} = 15$$

(2 marks) (d) Suppose the player plays this game 1000 times. Write an approximate expression for the probability that the *total* winnings are in the range $(14497.5, 15502.5]$. (Assume that 1000 is large enough for the Central Limit Theorem to hold.)

> **Solution:** The variance of W is
>
> $$\sigma^2(W) = \frac{(5/6) \cdot 3}{(1/6)^2} = 90$$

(1 mark for this computation.)

If W_i denotes the winning in the i-th game, then (by the Central Limit Theorem) the variable $T = \sum_i W_i$ follows a distribution *close* to the normal distribution with mean $E(T) = 1000 \cdot E(W)$ and variance $\sigma^2(T) = 1000 \cdot 90$. The "normalised variable" $X = (T - 15000)/\sqrt{90000}$ needs to take values in the range

$$\left(\frac{14497.5 - 15000}{300}, \frac{20002.5 - 15000}{300}\right] = (-1.675, 1.675]$$

Since the distribution of X is approximately the normal distribution, we have

$$P(-1.675 < X \leq 1.675) \approx \frac{1}{\sqrt{2\pi}} \int_{-1.675}^{1.675} \exp(-t^2/2) dt$$

(1 mark for this formula or any equivalent formula.)

(e) (Bonus Question) Two other players I. M'Patient and I. M'Greedy also play a similar game (against the same casino). However, I. M'Patient stops the game when there have been 15 non-1 throws even if less than three 1's have been seen.

On the other hand, I. M'Greedy refuses to take any winnings *less than* 15 rupees and gives them back to the casino. Which of these two players has higher expected winnings.

(4 marks) 2. Suppose that an experimental measurement results in values in the set $\{-1, 0, 3\}$ with probabilities $\{3/5, 1/5, 1/5\}$ respectively.

We repeat the experiment 300 times. Use Chebychev's inequality to give a lower bound for the probability that the average of these measurements is in the range $(-1/10, 1/10)$.

Solution: Let X_i denote the random variables which are the results of these experiments. We have $E(X_i) = -3/5 + 3 \cdot 1/5 = 0$. Hence, if $Y = \sum_i X_i/300$, then $E(Y) = 0$. (1 Mark for this calculation.)

We also calculate $\sigma^2(X_i) = 3/5 + 3^2 \cdot 1/5 = 12/5$. Hence, $\sigma^2(Y) = (12/5)/300 = 1/125$. (1 Mark for this calculation.)

We now apply Chebychev's inequality for Y

$$(1/10)^2 P(\mid Y \mid \geq (1/10)) \leq \sigma^2(Y)$$

This gives

$$P(-1/10 < Y < 1/10) \leq 1 - 100/125 = 1/5$$

(2 Marks for this step.)

3. The police car crosses the IISER Mohali traffic signal 36 times each day.

(1 mark) (a) Write an *expression* for the probability that no police car crosses for 15 minutes?

> **Solution:** We let W denote the random variable that measures the time in minutes to wait for a polic car to cross. Then, for $t \geq 0$, we have
>
> $$P(W > t) = e^{-at} \text{where} a = 36/(60 \cdot 24) = 1/40$$
>
> So the probability $P(W > 15) = e^{-3/8}$.

(1 mark) (b) Write an *expression* for the minimum time s we need to wait so that the probability of seeing a police car is at least 1/2.

> **Solution:** We want $P(W \leq s) \geq 1/2$. Now $P(W \leq s) = 1 - e^{-s/40}$, so we want
>
> $$e^{-s/40} \leq (1/2) \text{or} - s/40 \leq -\log(2)$$
>
> Thus $s \geq 40 \log(2)$.

(1 mark) (c) What is the expected amount of time that we need to wait to see a police car?

> **Solution:** This is
>
> $$E(W) = \int_0^\infty (t/40) \cdot e^{-t/40} dt$$
>
> We can calculate this directly or use the formula $E(W) = 1/a$ to get $E(W) = 40$.

4. In a question bank of 100 questions there are 7 questions that are really hard. Each of 190 students picks one question "at random" (same question can be picked by multiple students in principle).

(1 mark) (a) Write an expression for the probability that at most 5 students pick a hard question.

> **Solution:** The number H of hard questions picked in 190 choices is distributed according to the Binomial distribution with
>
> $$P(H = r) = \binom{190}{r} \left(\frac{7}{100}\right)^r \left(1 - \frac{7}{100}\right)^{190-r}$$
>
> Thus the answer is
>
> $$P(H \leq 5) = \sum_{r=0}^{5} \binom{190}{r} \left(\frac{7}{100}\right)^r \left(\frac{93}{100}\right)^{190-r}$$

(1 mark) (b) What is the expected number of hard questions picked?

Solution: The expectation $E(H)$ can be computed directly as in class or one can use the formula $E(H) = 190 \cdot (7/100) = 133/10$.

(1 mark) (c) Find a good approximation (in the form e^{-r} where r is a fraction) of the probability that no hard questions are picked.

Solution: Either by Poisson approximation or directly one sees that such an estimate is given by

$$\left(1 - \frac{7}{100}\right)^{190} \approx e^{-133/10}$$

Thus $r = 133/10$.