Boolean Algebra

1. Set operations on subsets of a fixed set S are defined as follows:

   - $A \wedge B$ is the collection of all elements of S that are in A as well as B.
   - $A \vee B$ is the collection of all elements of S that are either in A or in B.
   - A^c is the collection of all elements of S that are not in A.
   - $A \setminus B$ is the collection of all elements of S that are in A but not in B.

   Which of the following are correct identities for subsets A, B, C of S? Provide examples if they are not correct or prove equality if they are correct.

$$\begin{aligned}(A \vee B) \wedge C &= (A \wedge C) \vee (B \wedge C) \\ (A \wedge B) \vee C &= (A \vee C) \wedge (B \vee C) \\ (A \setminus B) \wedge C &= (A \wedge C) \setminus (B \wedge C) \\ (A \setminus B) \vee C &= (A \vee C) \setminus B \\ A \setminus B &= A \wedge B^c \\ (A \wedge B)^c &= A^c \vee B^c\end{aligned}$$

2. For a subset A of S let $\chi_A : S \to \{0, 1\}$ be the function whhich takes the value 1 on elements of A and 0 for elements of A^c. We define multiplication of elements of $\{0, 1\}$ as usual and addition by

$$0 \oplus 0 = 0; 0 \oplus 1 = 1 \oplus 0 = 1 \text{ and } 1 \oplus 1 = 0$$

   Write the following functions in terms of χ_A and χ_B.

   - χ_{A^c}
   - $\chi_{A \wedge B}$
   - $\chi_{A \vee B}$
   - $\chi_{A \setminus B}$

   What subsets corresponds to the functions given below?

   - $\chi_A \oplus \chi_B$
   - $1 \oplus \chi_A$.
   - $\chi_A \cdot (1 \oplus \chi_A)$.

   Write the condition that A is contained in B in terms of the functions χ_A and χ_B.

3. Assume that a class has students from each part of India (North/South/East/West), comprising of Boys and Girls, so that each of the letters of the English alphabet is the starting letter for exactly one student of each sex from each region. In other words, the information (Region, Sex, Starting Letter) uniquely specifies a student. Let A be the property that a student is from the North, B be the property that the student is a Boy, C be the property that the student's name starts the letter 'C' and D be the property that the student plays basketball for IISER Mohali. Explain the meaning of each set in the list below:
$$D^c; A \wedge B; A^c \wedge B^c \wedge D^c; D \setminus C$$

4. Suppose A, B and C are mutually exclusive events. Show (using only the laws of probability) that $P(A \vee B \vee C) = P(A) + P(B) + P(C)$.

5. Consider the set S of students in IISER Mohali. We pick a student at random (all students are equally likely to be chosen). Someone asserts that the probability that the student is from Hostel 8 is 0.3, and that the probability that it is a Male student is 0.6, and the probability that the student's name starts with 'Q' is 0.05. Finally, the person also says that the probability that it is not a Male student and not from Hostel 8 and that the name starts with 'A' is 0.7. Is it possible that all four estimates of probability are correct? If not, why not?

6. The probability that a car jumps the red light at Tribune Chowk is 0.01. The probablity that a car in Chandigarh has a white color is 99.9%. Is it possible that these are mutually exclusive events? Should the police issue registration only to white cars? Discuss!

7. If A and B are independent events, then show that A and B^c are independent events as well.

8. Four digits (0 to 9) are chosen at random (all digits are equally likely) in order to make a 4-digit number (which *may* start with 0). What is the probability that the number is divisible by 5? What is the probability that the number is divisible by 3?

   Now do the same problem *given* that the number does not start with 0.

9. (Starred) Three of us go out for ice-cream, but we have money only for two ice-creams. We don't want to share, so one of us has to do without ice-cream, but we want to do this fairly. Unfortunately, all we have is a coin which is fair. How do we we set up a game with a sequence of coin flips so that the probability that any one of us wins it is 1/3?