Solutions to Assignment 11

1. A box contains 3 coins C_1, C_2 and C_3 with probability of head as $1/2$, $2/3$ and $1/4$ respectively. You pick a coin out of the box but you don't know which one it is. You flip the coin 120 times. Justify your answer in each case below with paper and pencil estimates of the probabilities.
   (a) Suppose you get 70 heads. Which coin is the most likely to be the coin that you picked?
   (b) Suppose you get 90 heads. Which coin is the most likely to be the coin that you picked?
   (c) Suppose you get 40 heads. Which coin is the most likely to be the coin that you picked?
   (d) Suppose you get r heads. Is there a value of r (a positive integer!) for which it will be impossible to decide which coin it is?

Solution: Let $(p_1, p_2, p_3) = (1/2, 2/3, 1/4)$. We have

$$\begin{aligned} L(X=70;p_1)/L(X=70;p_2) &= \frac{3^{120}}{2^{190}} \\ L(X=70;p_2)/L(X=70;p_3) &= \frac{2^{310}}{3^{170}} \\ L(X=70;p_1)/L(X=70;p_3) &= \frac{2^{360}}{3^{50}} \end{aligned}$$

W can then simplify this

$$\begin{aligned} \frac{3^{120}}{2^{190}} &= \left(\frac{3^{12}}{2^{19}}\right)^{10} \\ \frac{2^{310}}{3^{170}} &= \left(\frac{2^{31}}{3^{17}}\right)^{10} \\ \frac{2^{360}}{3^{50}} &= \left(\frac{2^{36}}{3^{5}}\right)^{10} \end{aligned}$$

Now

$$\frac{2^{36}}{3^5} > \frac{2^{36}}{3^6} = (64/9)^6 > 1$$

and

$$\frac{2^{31}}{3^{17}} > \frac{2^{30}}{3^{18}} = \left(\frac{2^{10}}{3^6}\right)^3 = (1024/729)^3 > 1$$

The last expression is too close to do by paper and pencil alone! Instead we note (using a computer or with some hard work with a paper and pencil!)

$$\log(3) = log(3/2) - log(1/2) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)2^{2k}} > 1+1/12+1/180+1/448 = 7379/6720$$

and

$$\log(2) = -log(1/2) = \sum_{k=0}^{\infty} \frac{1}{k2^k} < 1/2+1/8+1/24+1/160+1/384+1/448 = 9319/13440$$

We then calculate

$$12 * \log(3) - 19 * \log(2) > 12 * (7379/6720) - 19 * (9319/13440) = 1/384 > 0$$

It follows that $(3^{12})/(2^{19}) > 1$. Hence, $L(X = 70; p_1)$ is the greatest.

The method for the next two parts is similar. In each case we calculate the ratios $L(X = r; p_i)/L(X = r; p_j)$ for (i, j) in $\{(1, 2), (1, 3), (2, 3)\}$. This will help us decide which of the three is the greatest. That is the most likely.

In order for $L(X = r; p_1)/L(X = r; p_2) = 1$ or $L(X = r; p_2)/L(X = r; p_3) = 1$ or $L(X = r; p_1)/L(X = r; p_3) = 1$ we need an identity like $2^a = 3^b$ for *integers* a and b. This is impossible unless $a = b = 0$.

It follows that there is no value of r for which the greatest amongst $L(X = r; p_i)$ cannot be found.

2. A child psychologist believes that the intelligence of all children can be classified into 4 categories according to the number of questions that they can answer. Category A answers 1/4 of all questions, Category B answers 1/2 of all questions, Category C answers 3/4 of all questions and Category D answers 9/10 of all questions. A certain child is tested on 100 questions.

   (a) Suppose the child answered 20 questions. Which category would the child be put in.

   (b) Suppose the child answered 40 questions. Which category would the child be put in.

   (c) Suppose the child answered 70 questions. Which category would the child be put in.

   (d) Suppose the child answered 80 questions. Which category would the child be put in.

   (e) Suppose the child answered 85 questions. Which category would the child be put in.

Solution: We make a take of the ratio of the likelihood of the child being in category A, B, C, D with respect to each other.

	$L(B)/L(A)$	$L(C)/L(A)$	$L(D)/L(A)$	$L(B)/L(C)$	$L(B)/L(D)$	$L(C)/L(D)$
20	$\frac{3^{80}}{2^{100}}$	$\frac{3^{20}}{2^{100}}$	$\frac{9^{20}}{5^{100}}$	3^{60}	$\frac{5^{100}3^{40}}{2^{100}}$	$\frac{5^{100}}{3^{20}2^{100}}$
40	$\frac{3^{60}}{2^{100}}$	$\frac{3^{40}}{2^{100}}$	$\frac{9^{40}}{5^{100}}$	3^{20}	$\frac{5^{100}}{3^{20}2^{100}}$	$\frac{5^{100}}{3^{40}2^{100}}$
70	$\frac{3^{30}}{2^{100}}$	$\frac{3^{70}}{2^{100}}$	$\frac{9^{70}}{5^{100}}$	3^{-40}	$\frac{5^{100}}{3^{110}2^{100}}$	$\frac{5^{100}}{3^{70}2^{100}}$
80	$\frac{3^{20}}{2^{100}}$	$\frac{3^{80}}{2^{100}}$	$\frac{9^{80}}{5^{100}}$	3^{-60}	$\frac{5^{100}}{3^{140}2^{100}}$	$\frac{5^{100}}{3^{80}2^{100}}$
85	$\frac{3^{15}}{2^{100}}$	$\frac{3^{85}}{2^{100}}$	$\frac{9^{85}}{5^{100}}$	3^{-70}	$\frac{5^{100}}{3^{155}2^{100}}$	$\frac{5^{100}}{3^{85}2^{100}}$

Using a computer we simplify

	$L(B)/L(A)$	$L(C)/L(A)$	$L(D)/L(A)$	$L(C)/L(B)$	$L(D)/L(B)$	$L(D)/L(C)$
20	> 1	< 1	< 1	> 1	> 1	> 1
40	< 1	< 1	< 1	> 1	> 1	> 1
70	< 1	> 1	< 1	< 1	< 1	> 1
80	< 1	> 1	> 1	< 1	< 1	> 1
85	< 1	> 1	> 1	< 1	< 1	< 1

We conclude as follows:

1. In this case B is the most likely.
2. In this case A is the most likely.
3. In this case C is the most likely.
4. In this case D is the most likely.
5. In this case C is the most likely.

3. In order to check a coin for bias, it was flipped a large number of times.

   (a) 10000 flips resulted in 4819 Heads. What is the most likely estimate for the probability p of a Head with this coin?

   (b) The length and number of "runs" of Heads were also counted and resulted in the following table.

length	0	1	2	3	4	5	6	7	8	9	10	11	12	13
number	2548	1239	623	329	154	75	34	16	9	5	3	1	1	1

   (Here length 0 means we got a Tail on the first flip.) What is the most likely estimate for the probability p of a Head with this coin?

Solution: The likelihood function for a coin with probability p of getting head is *proportional* to $p^a(1-p)^b$ when there are a heads and b tails. By putting the derivative with respect to p to 0, we see that the maximum is attained at $p = a/(a+b)$.

In the first case this becomes $p = 0.4819$.

In the second case, each times we see a run of length l there are l heads followed by 1 tail (to end the run). Thus, the number of heads is

$$0 \cdot 2548 + 1 \cdot 1239 + \cdots + 13 \cdot 1 = 4962$$

and the number of tails is

$$2548 + 1239 + \cdots + 1 = 5038$$

Thus, in this case we have $p = 0.4962$.

4. A Geiger counter clicks every time an alpha particle is detected. Suppose the gaps between the clicks are (in seconds) given as 6.43, 1.21, 10.41, 4.32, 3.70, 0.55, 5.92. Give an estimate for the frequency of emission of an alpha particle by the radiation source. Justify that this is the most likely value for the frequency.

Solution: We assume that the emission follows a waiting time distribution with density $c\exp(-ct)$.

Assuming a least count of $e = .01$ we see that the probability of seeing a gap of T seconds is $\int_{T-e/2}^{T+e/2} c\exp(-ct)$ which is approximately $ec\exp(-cT)$.

Thus, the likelihood for the events seen is given by $e^n c^n \exp(-c\tau)$, where n is the number of events and τ is the sum of the waiting times.

Differentiating with respect to the parameter c and putting the derivative to 0, we solve for c to get $c = n/\tau$. In our case we have $n = 7$ and $\tau = 32.54$. This gives $c \simeq 0.215$.

5. A certain physics experiment results in a random variable X which has probability density $cf(ct)$ in the range $[-1/c, 1/c]$ (and 0 outside) where f is a non-negative differentiable function such that $\int_{-1}^{1} f(t)dt = 1$.

   The value of c is a universal constant but it is not known. Will it be possible to estimate the value of c by doing a large number of experiments? If so, try to find an estimator function.

Solution: *Assume* that f is continuous. As discussed in class and the notes, we can approximate the probability of a given observation $X \in [T - e/2, T + e/2]$, when e is the least count of the apparatus, as $ecf(cT)$. This requires the additional assumption that for the relevant values of c, $f(ct)$ does not vary much over intervals of size e.

Carrying out N experiments, we would obtain the probability of the result as

$$e^N c^N \prod_{i \leq N} f(cT_i)$$

Assuming that f is differentiable, we can take the derivative of the above expression with respect to c and put that equal to 0 to solve for c. This gives an equation of the form

$$\frac{N}{c} + \sum_{i \leq N} \frac{T_i f'(cT_i)}{f(cT_i)} = 0$$

Assuming that we can solve this equation to find a *unique* maximum, we can write an estimator for c as a function of the experimental results $T_1, \ldots, T_N$.

It is important to note that there are a number of assumptions involved in order to make this work.