headandfoot

## Solutions to Assignment 1,2,3,4,9

1. (Assignment 1; 2(c)) In the pair of numbers below, which is larger? Explain why. $n^5$, $10000 \cdot (n+1)^2$ for large $n$.

> **Solution:** Assignment 1 was only supposed to use basic arithmetic properties of numbers.
>
> We note the high-school expansion gives, for $n \geq 1$,
>
> $$(n+1)^2 = n^2 + 2n + 1 \leq n^2 + 2n^2 + n^2 \leq 5n^2$$
>
> Thus, $50000 \cdot n^2 > 10000 \cdot (n+1)^2$.
>
> It is clear that for $n > 50000$, we have $n^3 > 50000 \cdot n^2$. Since $n^5 \geq n^3$ for $n \geq 1$, we see that $n^5 > 10000 \cdot (n+1)^2$ for $n > 50000$.

2. (Assignment 1; 3(d)) Give two positive rational numbers $p/q$ and $r/s$ (this means that $p$, $q$, $r$ and $s$ are natural, or counting, numbers). Suppose that $p/q < r/s$. Order the above three numbers.

   1. $\frac{(p/q)+(r/s)}{2}$
   2. $\sqrt{(pr)/(qs)}$
   3. $(p+r)/(q+s)$

> **Solution:** Putting $a = p/q$ and $b = r/s$, we have $(a-b)^2 \geq 0$. This gives $a^2 + b^2 \geq 2ab$. Adding $2ab$ to both sides, we have $(a+b)^2 \geq 4ab$. Diving by 2 and taking (positive) square root, we get
>
> $$\frac{a+b}{2} \geq \sqrt{ab}$$
>
> In other words
>
> $$\frac{(p/q)+(r/s)}{2} \geq \sqrt{(pr)/(qs)}$$
>
> This means that the first number is not less than the second number.
>
> We note that,
>
> $$\frac{(1/3)+(1/2)}{2} = \frac{5}{12} > \frac{2}{5} = \frac{1+1}{3+2}$$
>
> on the other hand
>
> $$\frac{(1/2)+(2/3)}{2} = \frac{7}{12} < \frac{3}{5} = \frac{1+2}{2+3}$$

Hence, the first and last numbers cannot be compared in general.

Similarly,

$$\sqrt{(1/1)\cdot(1/4)} = \frac{1}{2} > \frac{2}{5} = \frac{1+1}{1+4}$$

On the other hand

$$\sqrt{(4/1)\cdot(1/1)} = \frac{2}{1} < \frac{5}{2} = \frac{4+1}{1+1}$$

Hence the last two numbers cannot be compared in general.

3. (Assignment 1; 4(c)) Given that $p$ and $q$ are counting numbers so that $p^2 > 3q^2$ and put $r/s = (2p+3q)/(p+2q)$. Show that:

   - $r^2 > 3s^2$
   - $r/s < p/q$

   Use *this* idea to find a rational number $a/b$ so that $100(a^2 - 3b^2) < b^2$.

**Solution:** We note that $3^2 > 3 \cdot 1^2$ so we start with $p_1/q_1 = 3/1$ and define

$$\frac{p_{n+1}}{q_{n+1}} = \frac{2p_n + 3q_n}{p_n + 2q_n}$$

This gives us the sequence

$$\frac{3}{1}, \frac{9}{5}, \frac{33}{19}, \frac{123}{71}$$

We check that the last fraction $a/b$ satisfies the condition $100(a^2 - 3b^2) < b^2$.

4. (Assignment 2; 1(e)) Compare the following sequences to decide which one is *eventually* larger. The sequence with general term $2^n$

$$2^1, 2^2, 2^3, \ldots$$

   versus the Fibonacci sequence with general term $F(n) = F(n-1) + F(n-2)$ starting with $F(1) = 5$ and $F(2) = 8$.

$$5, 8, 13, \ldots$$

**Solution:** We note that $F(1) < F(2)$ and claim, by induction, that $F(n) < F(n+1)$ which we already have for $n = 1$. Let assume that we have proved that $F(1) < F(2) < \cdots < F(n) < F(n+1)$. Then

$$F(n+1) = F(n) + F(n-1) < F(n+1) + F(n) = F(n+2)$$

Hence, we have proved, by induction that $F(n) < F(n+1)$.

Now

$$F(n) = F(n-1) + F(n-2) < F(n-1) + F(n-1) = 2F(n-1)$$

So, if we find *one* $k$ so that $F(k) < 2^k$, then $F(k+n) < 2^{k+n}$ for all $n \geq 0$, which we can prove by induction on $n$ starting with $n = 0$.

We check that $F(6) = 2 \cdot 8 + 3 \cdot 13 = 55$ while $2^6 = 64$. So we have a candidate $k$.

5. (Assignment 2; 2(e)) Give the properties of the sequence out of:

$$\text{eventually increasing, eventually decreasing, neither, bounded, unbounded}$$

The sequence with general term $n \cdot \sin(1/n)$

$$\sin(1), 2\sin(1/2), 3\sin(1/3), \ldots$$

**Solution:** Since the function sin was defined later, this makes use of things not proved about this function at this point in the course! Here is a proof that does not use calculus.

The function sin is defined by the series

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

which converges uniformly and absolutely in $|x| \leq r$ for all $r$. It follows that

$$\frac{\sin(x)}{x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!}$$

For $x = 1/n$ we get an alternating series $\sum_{k=0}^{\infty} (-1)^k a_{k,n}$ with

$$a_{k,n} = \frac{1}{n^{2k}(2k+1)!}$$

Since the series is alternating, it follows that $n\sin(1/n)$ lies between

$$p_n = \left(1 - \frac{1}{n^2 3!}\right) \text{ and } q_n = \left(1 - \frac{1}{n^2 3!} + \frac{1}{n^4 5!}\right)$$

Now,

$$p_{n+1} - p_n = \frac{1}{3!}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \frac{2n+1}{n^2(n+1)^2 3!}$$

For large $n$, it follows that

$$p_{n+1} - p_n > q_n - p_n = \frac{1}{n^4 5!}$$

In other words, for large $n$ the ordering is

$$1 > q_{n+1} > p_{n+1} > q_n > p_n$$

It follows that $(n+1)\sin(1/(n+1)) > n\sin(1/n)$ is an increasing sequence.

The argument using calculus is a bit simpler. We prove that (using the power series) $f(x) = \sin(x)/x$ is twice continuously differentiable at $x = 0$.

1. Its derivative $f'(x)$ takes the value 0 at $x = 0$.
2. Its second derivative $f''(0)$ is $-1/6$.

From these three properties, we show that $f'(x)$ is negative for positive values of $x$ near 0. From this it follows that $\sin(x)/x$ decreases with increasing $x$ near $x = 0$. Thus $n\sin(1/n)$ is increasing with increasing $n$.

6. (Assignment 2; 3(c)) Does the following sequence have an *upper* bound? The sequence with general term $1 + 1/2 + \cdots + 1/n!$ where $n! = 1 \cdot 2 \cdots n$ is the factorial of $n$.

**Solution:** It has been shown in the notes that this is bounded above.

7. (Assignment 3; 1(g)) Does the following series converge or does it diverge to infinity? The series $\sum_{n=1}^{\infty} n \cdot x^n$ for $0 < x < 1$.

**Solution:** This too has been shown in one of the assignments. The key point is that the condition $0 < x < 1$ *must* be used to ensure that the sum of the infinite geometric series makes sense.

8. (Assignment 4; 1(f) and 1(g)) Find the limit superior and the limit inferior of the following sequences.

(1 (bonus)) (a) For each $n$, let $k_n$ be such that $2^{k_n}$ is the *smallest* power of 2 which is greater than $n$; in other words $2^{k_n-1} \leq n < 2^{k_n}$. Now take the sequence $(n/2^{k_n})_{n\geq 1}$.

**Solution:** For $k \geq 1$, if $n = p_k = 2^k - 1$, then it is clear from the definition that $k_n = k$. Moreover, $p_r > p$ (prove it by induction!). Thus

$$\sup \left(\frac{n}{2^{k_n}}\right)_{n\geq r} \geq \sup \left(1 - \frac{1}{2^k}\right)_{k\geq r} = 1$$

On the other hand it is clear that $n/2^{k_n} < 1$. This proves that

$$\sup \left(\frac{n}{2^{k_n}}\right)_{n\geq r} = 1$$

for *all* $r$. Hence $\limsup(n/2^{k_n}) = 1$.

For $k \geq 1$, if $n = q_k = 2^{k-1}$, then it is clear from the definition that $k_n = k$. Again, $q_{r+1} > r$ (prove it by induction). Thus

$$\inf \left(\frac{n}{2^{k_n}}\right)_{n\geq r} \leq \inf \left(\frac{2^{k-1}}{2^k}\right)_{k\geq r+1} = \frac{1}{2}$$

On the other hand it is clear that $1/2 \leq n/2^{k_n}$ for all $n$. This proves that

$$\inf \left(\frac{n}{2^{k_n}}\right)_{n\geq r} = 1/2$$

for *all* $r$. Hence $\liminf(n/2^{k_n}) = 1/2$.

(1 (bonus)) (b) $(\sin(n))_{n\geq 1}$.

**Solution:** This is a more subtle result than what has been proved during this course.

To explain this, we first need some notation. Given any positive number $x$, by the Archimedean principle, there is a non-negative integer $n$ so that $x < n + 1$. Let $[x]$ denote the smallest such integer, called the "integer part" of $x$. Then, we have $[x] \leq x < [x] + 1$. We further denote by $\{x\} = x - [x]$ and call it the "fractional part" of $x$. Note that we have

$$x = [x] + \{x\} = \text{integer part of } x \ + \ \text{fraction part of } x$$

for *any* number $x$.

We need two statements, one of which is *very* difficult to prove:

1. Let $\alpha$ be an irrational number. Given *any* number $r$ in $[0,1]$, there is a sequence $n_1 < n_2 < \cdots$ of positive integers such that $(\{n_k\alpha\})_{k\geq 1}$ converges to $r$.

2. The number $\pi$ is irrational.

Assuming these results, and the properties of $\sin(x)$ proved in the notes, we can proceed as follows.

Take $\alpha = 1/(2\pi)$ and $r = 1/4$. By the above statement, we can find an increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that the sequence $(\{n_k/(2\pi)]\})_{k\geq 1}$ converges to $1/4$. Now,

$$\frac{n_k}{2\pi} = \left[\frac{n_k}{2\pi}\right] + \left\{\frac{n_k}{2\pi}\right\}$$

Multiplying by $2\pi$ we have

$$n_k = (2\pi)\left[\frac{n_k}{2\pi}\right] + (2\pi)\left\{\frac{n_k}{2\pi}\right\}$$

By the properties of sin proved in the notes we then obtain

$$\sin(n_k) = \sin\left((2\pi)\left\{\frac{n_k}{2\pi}\right\}\right)$$

By the properties of sequences proved in the notes

$$\lim\left((2\pi)\left\{\frac{n_k}{2\pi}\right\}\right)_{k\geq 1} = (2\pi)\lim\left(\left\{\frac{n_k}{2\pi}\right\}\right)_{k\geq 1}$$

By assumption the second limit is $(2\pi)(1/4) = \pi/2$. By the continuity of sin we get

$$\lim(\sin(n_k)_{k\geq 1}) = \sin\left((2\pi)\lim\left(\left\{\frac{n_k}{2\pi}\right\}\right)_{k\geq 1}\right) = \sin(\pi/2)$$

In the notes we have proved that $\sin(\pi/2) = 1$. Thus, we have proved that $\lim(\sin(n_k))_{k\geq 1} = 1$. As proved in the notes $|\sin(x)| \leq 1$ for all $x$. It follows easily that $\limsup(\sin(n))_{n\geq 1} = 1$.

Choosing $r = 3/4$ instead of $1/4$ in the above argument we can also find a increasing sequence $(m_k)_{k\geq 1}$ of positive integers so that $\lim(\sin(m_k))_{k\geq 1} = \sin(3\pi/4) = -1$. This then gives $\liminf(\sin(n))_{n\geq 1} = -1$.

We will not attempt to prove that $\pi$ is irrational. This is a theorem somewhat beyond what can be easily achieved in these notes. The other statement assumed above *can* be proved and is left as an interesting (not easy!) exercise.

9. (Assignment 9; 1(c)) The following question is about continuous functions on an interval

$[a, b]$.

Find an example of non-zero functions $f$ and $g$ on $[0, 1]$ such that $f \cdot g = 0$.

**Solution:** We define

$$f(x) = \begin{cases} x - 1/2 & x \leq 1/2 \\ 0 & x \geq 1/2 \end{cases}$$

and

$$g(x) = \begin{cases} x - 1/2 & x \geq 1/2 \\ 0 & x \leq 1/2 \end{cases}$$

It is clear that $f(x)g(x) = 0$ for all $x$ lying in $[0, 1]$. Moreover, $f$ and $g$ are non-zero. It is clear that $f$ and $g$ are continuous on $[0, 1/2)$ and $(1/2, 1]$. It remains to check that they are contiuous at 0 and that is easily done.