Solutions

1. For each of the following statements provide a justification in two or three sentences. Overly long and complex answers will lead to *deduction* of marks.

   The exercises were *supposed* to introduce and prove the basic properties of $\tan(x)$ and its inverse function. Hence it was perturbing to find many people *using* such properties of this function as "standard".

   The basic idea of the course has *always* been that you should only use what you can *prove*. To say you *know* something in Mathematics, but don't know its proof is *just plain wrong*!

   - You can quote a result or definition from the Moodle notes.
   - You can quote some standard result or definition by name; for example, the "addition law for the sine function". This was meant to provide an avenue for people who had read some other books and have read proofs there. Not to allow wholesale import of *all* results in mathematics.
   - In each part you can use one of the earlier parts (but not a later part). You were supposed to use the *statement* of an earlier part, not your (or anyone else's!) proof of it. Many people referred to the proof of an earlier part.

   Try to do the questions in order.

(2 marks) (a) If $(a_n)_{n\geq 0}$ is a convergent sequence, and $b_n = a_{n+1} - a_n$, then $(b_n)_{n\geq 1}$ converges to 0.

> **Solution:** For every k positive integer, there is a positive integer p so that $|a_n - a_m| < 1/k$ for $n, m \geq p$. Since $b_n = a_{n+1} - a_n$, it follows that $|b_n| < 1/k$ for $n \geq p$ as required.

(2 marks) (b) If $\sum_{n=0}^{\infty} a_n$ is a convergent series then the sequence $(a_n)_{n\geq 0}$ converges to 0.

> **Solution:** Convergence of the series is the same as the convergence of the sequence $(s_n)_{n\geq 0}$ of partial sums $s_n = \sum_{k=0}^{n} a_n$. Morover $a_{n+1} = s_{n+1} - s_n$, so the result follows from the previous part (a).

(2 marks) (c) There is a sequence $(a_n)_{n\geq 0}$ which converges to 0, but the series $\sum_{n=0}^{\infty} a_n$ *does not* converge.

> **Solution:** We can take $a_n = 1/(n+1)$. As seen in the notes $\sum_{n=0}^{\infty} a_n$ diverges.

(2 marks) (d) *If* $\sum_{n=0}^{\infty} (-1)^n a^{2n}$ converges for some number a, *then* $|a| < 1$.

Solution: Applying the previous part (b), the sequence $(|x^{2n}|)_{n\geq 0}$ must converge to 0. If $|x| \geq 1$, then the sequence consists of numbers ≥ 1, and so it cannot converge to 0.

(2 marks) (e) The series $s(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ converges absolutely and uniformly for $|x| \leq r$ and $0 < r < 1$.

Solution: The radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is $R > 0$ if $\limsup |a_n|^{1/n}$ is finite and equal to $1/R$. For this particular series we calculate this limit superior to be 1.

Alternatively, one can use the convergence of Geometric series which was covered in the notes.

(2 marks) (f) The function $s(x)$ is continuously differentiable for x in an interval of the form $|x| \leq r$ for $0 < r < 1$.

Solution: Since the series converges uniformly and absolutely for this region by part (e), it defines a continuously differentiable function in this region.

(2 marks) (g) For $|x| < 1$, the product

$$(1 + x^2) \cdot s(x) = 1$$

is equal to 1.

Solution: We can treat $1 + x^2 = \sum_{n=0}^{\infty} a_n x^n$ as a power series with $a_0 = 1 = a_2$ and $a_n = 0$ for all other n; it converges uniformly and absolutely in intervals $|x| < r$ for all positive values of r. By the product of power series, we see that the coefficients of product power series are given by $c_0 = 1$ and for $n \geq 1$,

$$c_{2n} = a_0(-1)^n + a_2(-1)^{n-1} = 0$$

and $c_{2n-1} = 0$. In other words, the product power series is the power series of the constant function 1.

Alternatively, one can use the Geometric series to calculate the partial sums and prove convergence.

(2 marks) (h) If f a continuous positive function on $[a, b]$, and r a fixed number, then $g(x) = \exp(r \log(f(x)))$ is also a continuous function for x in the interval $[a, b]$.

Solution: The composition of continuous functions is also continuous wherever it is defined. The function log is defined on $[0, \infty)$ and the function exp is defined for the whole real line. Since the function f is positive on $[a, b]$, the fucntion g is well-defined on this interval.

(2 marks) (i) If f a continuously differentiable positive function on $[a, b]$, and r a fixed number, then the function $g(x) = \exp(r \log(f(x)))$ is differentiable for all x in the interval $[a, b]$ and this derivative $g'(x)$ satisfies

$$g'(x) = g(x) \cdot \frac{rf'(x)}{f(x)}$$

where $f'(x)$ is the derivative of f at x.

> **Solution:** The composition of continuously differentiable functions is also continuously differentiable wherever it is defined; we have already checked in the previous part (h) that this is well-defined. The chain rule gives the derivative as
>
> $$g'(x) = \exp(r \log(f(x))) \cdot r \frac{1}{f(x)} \cdot f'(x)$$
>
> which yields the required formula.

(2 marks) (j) If f is a continuously differentiable positive function on the interval $[a, b]$, then $g = 1/f$ is a continuously differentiable positive function on the interval $[a, b]$ and its derivative $g'(x)$ at x is

$$g'(x) = -\frac{f'(x)}{f(x)^2}$$

> **Solution:** We apply the previous part (i) to the case $r = -1$. Alternatively, one can apply the Chain rule to $g = i \circ f$ where $i(x) = 1/x$. The latter is a continuously differentiable function for $x > 0$.

(2 marks) (k) For any two real numbers a and b such that $a < b$, there is a continuously differentiable function t on $[a, b]$ such that $(1 + x^2) \cdot t(x) = 1$ for all x in the interval $[a, b]$.

> **Solution:** Apply the previous part (j) with $f(x) = 1 + x^2$ to get $t(x) = \exp(-\log(f(x)))$. The function f is positive (in fact ≥ 1) on any interval $[a, b]$. By the homomorphism property of exp and log we see that $f(x) \cdot t(x) = 1$ for all x.

(2 marks) (l) We have $t(x) = s(x)$ *only* for x in the region $|x| < 1$.

> **Solution:** Clearly $t(x) = s(x) = 1/(1 + x^2)$ for x in this region. By a previous part, the power series does not converge for $|x| \geq 1$. So $s(x)$ is only *defined* in this region.

(2 marks) (m) There is a continuously differentiable function A defined on the whole real line such that $A(0) = 0$ and the derivative $A'(x)$ of A at a point x satisfies $A'(x) = t(x)$. We will use this function A in later parts.

> **Solution:** The function $t(x)$ is continuous on the whole real line. By the fundamental theorem of calculus the function
>
> $$A(x) = \int_0^x t(u)du$$
>
> has the required properties.

(2 marks) (n) The power series

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

converges to $A(x)$ for x in the region $|x| < 1$.

> **Solution:** By earlier parts (e) and (l), the power series $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ converges uniformly and absolutely to $t(x)$ for x in $|x| \leq r$ for $0 < r < 1$. Hence, we can integrate term by term (as proved in the notes) to obtain the desired formula.

(2 marks) (o) The function A defined above is strictly monotonically increasing.

> **Solution:** Since $A'(x) = t(x)$ is positive and continuous, this follows from results in the notes.

(2 marks) (p) We have the inequality $A(n + 1) - A(n) \leq 1/(1 + n^2)$ for all positive integers n.

> **Solution:** Since $A'(x) = t(x)$, and $t(x) \leq 1/(1 + n^2)$ for x in the interval $[n, n + 1]$,
>
> $$A(n+1) - A(n) = \int_n^{n+1} t(x)dx$$
> $$\leq \int_n^{n+1} \frac{1}{1+n^2}dx = \frac{1}{1+n^2}$$
>
> Alternatively, by the Mean Value Theorem, there is a c in $[n, n + 1]$ such that $A(n + 1) - A(n) = A'(c)(n + 1 - n)$. It follows that
>
> $$A(n + 1) - A(n) = t(c) \leq 1/(1 + n^2)$$

(2 marks) (q) The sequence $(A(n))_{n\geq 1}$ is bounded.

Solution: We apply the previous part (p) to get

$$A(n) = A(1) + \sum_{p=1}^{n-1} (A(p+1) - A(p))$$
$$\leq A(1) + \sum_{p=1}^{n-1} \frac{1}{1+p^2} \leq A(1) + \sum_{p=1}^{n-1} \frac{1}{p^2}$$
$$\leq A(1) + \sum_{p=1}^{\infty} \frac{1}{p^2}$$

We have seen in the notes that the series $\sum_{p=1}^{\infty}(1/p^2)$ is convergent.

(2 marks) (r) There is a number μ such that $A(x) < \mu$ for all positive numbers x and μ is the smallest such number.

Solution: Let μ be the least upper bound of the increasing sequence $(A(n))_{n\geq 1}$. By the Archimedean principle, given x, there is a positive integer n so that $x < n$. Since f is increasing, $A(x) < A(n) < \mu$. By the definition of least upper bound, μ is the smallest such number.

(2 marks) (s) Let τ be the smallest positive number x such that $\cos(x) = 0$ (as defined in the notes). Then $\sec(x) = 1/\cos(x)$ is a continuously differentiable function in the region $0 \leq x < \tau$ and its derivative is

$$\frac{d}{dx}\sec(x) = \frac{\sin(x)}{\cos(x)^2}$$

Solution: As proved in the notes $\tau > 0$. So, for $0 < y < \tau$, the function $\cos(x)$ is positive in the interval $[0, y]$. It follows from part (j) that $1/\cos(x)$ is differentiable and its derivative is $\sin(x)/\cos(x)^2$ by the chain rule.

(2 marks) (t) Let τ be as above. Then $\tan(x) = \sin(x) \cdot \sec(x)$ is a continuously differentiable function in the region $0 \leq x < \tau$ and its derivative is

$$\frac{d}{dx}\tan(x) = 1 + \tan(x)^2$$

Solution: By the product rule we have

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\sin(x)\cdot\sec(x) + \sin(x)\cdot\frac{d}{dx}\sec(x)$$
$$= \cos(x)\cdot\frac{1}{\cos(x)}\sin(x)\cdot\frac{\sin(x)}{\cos(x)^2} = 1+\tan(x)^2$$

as required.

(2 marks) (u) For τ as above we have $\tan(\tau/2) = 1$.

Solution: By the addition laws for Cosine, we have

$$0 = \cos(\tau/2+\tau/2) = \cos(\tau/2)^2 - \sin(\tau/2)^2$$

We have seen in the notes that $\sin(x)$ and $\cos(x)$ are positive for $0 < x < \tau$. So $\cos(\tau/2) = \sin(\tau/2)$ since each positive number has a unique positive square root as proved in the notes.

(2 marks) (v) Let g be a continuously differentiable function on some interval of the form $[0, y]$ such that $g(0) = 0$ and $g'(x) = 1$ for all x in the interval. Then $g(x) = x$ for all x in the interval.

Solution: By the fundamental theorem of calculus, we have

$$g(x) = \int_0^x dt$$

We have seen in the notes that the latter integral is x.

(2 marks) (w) We have the identity $A(\tan(x)) = x$ for x in the region $0 \le x < \tau$.

Solution: By the Chain rule, the composite function is differentiable and its derivative is

$$\frac{d}{dx}A(\tan(x)) = \frac{1}{1+\tan(x)^2}\cdot\left(1+\tan(x)^2\right) = 1$$

We also have $A(\tan(0)) = A(0) = 0$. Now apply the previous part.

2. We could try to use all the above parts to prove the following statements. However, in each case some step/statement is missing. What is it?

(2 marks) (a) We have the equality $\mu = \tau$.

> **Solution:** We need to prove that there is a sequence $(x_n)_{n\geq 1}$ of numbers less than τ such that $\tan(x_n) > n$. It will then follow that $f(\tan(x_n))$ converges to μ.

(2 marks) (b) We have the equality

$$\frac{\tau}{2} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

> **Solution:** From (l) we know that for $0 \leq x < 1$, we have
>
> $$A(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$
>
> We need to extend this to $x = 1$ by estimating the error terms.

Some common errors

1. You cannot do various operations "term-by-term" for a series without providing justification. For example, differentiating terms, grouping terms, cancelling terms from different series. For each of these, one can give examples (in fact some examples are some of the question parts!) where this operation fails *in general*. So you must give a reason why it works in your case.

2. You cannot differentiate (or integrate) complicated expressions without providing justification of the technique used. For example, product rule, chain rule etc.

3. Words like "closer", "larger", "eventually" are fine for informal discussions. In a mathematics paper you need to make these ideas precise by providing bounds.

4. There is a reason that proofs in mathematics follow a sequence. A result should not be proved by using it in a round-about way. That is why the exercises were in sequence. Using a later part in an earlier part was not allowed.

5. There is no theorem that says that the Taylor series of a function determines the function unless one gives *additional* conditions on the function. This was pointed out in class during the discussing of $\exp(-1/x^2)$ which can be added to any Taylor series to "spoil the show".